Evolution of linear gravitational and electromagnetic perturbations inside a Kerr black hole

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(Received 12 February 1999; published 9 December 1999)

We analyze the evolution of linear gravitational ($s=\pm 2$) and electromagnetic ($s=\pm 1$) perturbations inside a Kerr black hole, within the framework of the Newman-Penrose formalism. In particular, we derive explicit expressions for the asymptotic behavior of the perturbations at the early portion of the Cauchy horizon (CH). The calculation is carried out in the time domain, using late-time expansion. The initial data are the presumed inverse-power tails at the event horizon. We find that the ''outgoing'' fields $s<0$ are regular (though nonvanishing) at the CH. However, the ''ingoing'' fields $s>0$ diverge at the CH-like $(r-r_-)^{-s}$, where r is the radial Boyer-Lindquist coordinate and r_- is its value at the CH. This divergent term is multiplied by an inverse power of the ingoing Eddington coordinate v. For nonaxially symmetric modes $(m\neq 0)$, the divergence of the $s>0$ fields is also modulated by an oscillatory term $e^{im\Omega_- v}$, where Ω_- is a fixed parameter and m is the magnetic number of the mode under consideration. This term exhibits an infinite number of oscillations on the approach to the CH. We also find that the nonaxially symmetric modes diverge faster than the axially symmetric ones. Based on the result of a previous nonlinear perturbation expansion, which showed that the nonlinear perturbation terms are negligible at the CH compared to the linear ones, we argue that the linear gravitational perturbations calculated here correctly describe the strength and features of the curvature singularity at the CH (to the leading order in $1/v$ and $1/u$).

PACS number(s): 04.70.Bw

I. INTRODUCTION

Kerr geometry [1] is a unique solution of the vacuum Einstein equations describing a stationary spinning black hole (BH). The study of spinning BHs has a strong motivation, because realistic astrophysical BHs are expected to be rapidly rotating [2,3]. The rotation has a dramatic effect on the inner structure of the BH: Without the rotation, the (static) BH is described by the Schwarzschild geometry, and its interior is sealed by an all-encompassing spacelike singularity. On the contrary, in the Kerr geometry there is no such spacelike singularity. Instead, there is a null hypersurface inside the BH, known as the inner horizon or Cauchy horizon (CH) (see Fig. 1). This null hypersurface, which is a perfectly smooth place in pure Kerr geometry, has drastic implications to the causal structure of the black hole: The CH is the boundary of the domain of dependence of a spacelike hypersurface S (see Fig. 1) in the external universe; hence, it is the boundary of predictability for physical fields evolving from initial data specified on S.

It is remarkable that in a Kerr BH predictability breaks down by an approach to a regular CH, and not by a destruction due to divergent tidal forces at a curvature singularity. However, Penrose [4] pointed out more than 30 years ago that the CH is a hypersurface of infinite blueshift. That is, soft photons falling from the external universe into the BH become infinitely blueshifted (as measured by a typical infalling observer moving towards the CH) when the CH is approached. This infinite blueshift results from the mapping (along radial ingoing null geodesics) of infinite time intervals in the external universe into a finite time interval near the CH. Penrose pointed out that this geometrical-optics phenomenon of unbounded blueshift strongly suggests the instability of the CH to small gravitational or electromagnetic perturbations.

The Reissner-Nordström (RN) solution, describing the geometry of a static, spherical, charged BH, has an inner structure similar to the Kerr solution. In particular, it has a CH which too is a surface of infinite blueshift. (In fact, Penrose's original argument [4] was primarily related to the CH in the RN geometry.) The instability of the CH of the RN BH to linear electromagnetic and gravitational perturbations was verified by several authors [5–8]. This instability suggests that in a more realistic BH model the regular CH will be

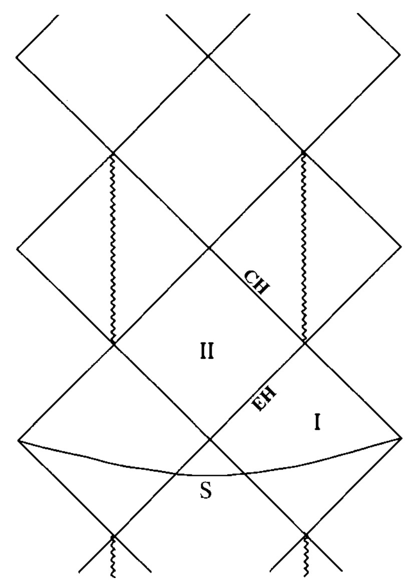

FIG. 1. Penrose diagram of the extended Kerr geometry. The event and Cauchy horizons are marked EH and CH, respectively. The asymptotic region I is the external universe. The present paper deals with region II, i.e., the black-hole interior between the EH and CH. The curve S represents a typical initial hypersurface outside the black hole.

0556-2821/99/61(2)/024001(20)/$15.00

significantly modified—presumably it will be converted into a curvature singularity of some type. This prediction was verified analytically [9–13] and numerically [14–16] for a class of nonlinear spherical toy models made of a charged BH perturbed by either null fluids or a self-gravitating scalar field. In these nonlinear models the regular CH is converted into a curvature singularity which is strictly null [9–10] and weak [11] (we use here Tipler's [17] notion of weakness).

Realistic astrophysical BHs, however, are spinning (and uncharged). In general, one expects the situation in a spinning BH to be similar in many respects to that in spherical charged BHs. In the spinning case, too, the CH is a hypersurface of infinite blueshift, suggesting a linear instability. Indications for such a linear instability of the Kerr background were found by McNamara [18] and by Novikov and Starobinsky [19]. In the fully nonlinear context, this instability suggests that small perturbations will convert the regular CH into a curvature singularity. This expectation was verified by a nonlinear perturbation analysis of the interior of a Kerr BH [20]. This analysis revealed the asymptotic behavior of metric perturbations near the CH, both in the linear context and in the context of fully nonlinear perturbations (within the framework of a systematic nonlinear perturbation expansion). The regular CH of pure Kerr geometry was indeed found to be converted, in the presence of gravitational perturbations, into a spacetime singularity, marked by the divergence of curvature. This curvature singularity, to which we shall refer as the *CH singularity*, was found to be strictly null, weak, and scalar curvature (namely, the curvature scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ diverges) [20]. The weakness of the singularity means that the actual tidal distortion experienced by an infalling physical object is bounded—and in fact negligibly small for a wide range in the space of geodesics—when the curvature singularity at the CH is approached. The analysis in Ref. [20] also suggests that for nonaxially symmetric modes the divergence of curvature at the CH of a spinning BH is oscillatory. Namely, the nonaxially symmetric metric perturbations exhibit an infinite number of oscillations on the approach to the CH [21]. As a consequence, the curvature is expected to exhibit such oscillations too. (This is to be compared with the monotonous mass-inflation singularity [10,11] inside spherical charged BHs.)

Recently, the structure of the CH singularity was examined in a nonperturbative leading-order analysis by Brady *et al.* [22]. This analysis provided a description of the singularity in terms of geometric entities like the shear, expansion, and twist of null congruences intersecting the CH, and the Newman-Penrose scalars of the Weyl tensor. The analysis in Ref. [22] is local; namely, it is based on a local characteristic initial-value setup near the CH (assuming inverse-power local initial data). The calculation is carried out to the leading order in the inverse blueshift factor $e^{-\kappa_- v}$ (κ_- and v are defined below), which is essentially the affine distance from the CH. Nevertheless, it handles the nonlinearity of the Einstein equations in a fully nonperturbative manner. This analysis confirmed the main findings of the nonlinear perturbation analysis [20], namely, that the singularity at the CH is blueshift dominated, null, weak, and scalar curvature.

Several rigorous nonperturbative local analyses [23–25] were also carried out in order to examine the local structure of the vacuum curvature singularity at the CH. Brady and Chambers [23] solved the constraints equations along two intersecting null hypersurfaces (one of which coincides with the CH). Later, Ori and Flanagan [24] handled both the constraints and the evolution equations, and mathematically constructed a class of vacuum solutions with a null, weak, scalar-curvature singularity. This class depends on a sufficient number (8) of quite arbitrary (though analytic) initial functions of the three spatial variables, so it may be regarded as generic. More recently, Ori [25] showed that there exists a generic class of null weak singularities within the family of linearly polarized plane-symmetric vacuum spacetimes. All these analyses [22–25] support the main result of Ref. [20] (concerning the *local* behavior near the CH), i.e., the existence and genericity of vacuum solutions with a strictly null, weak, scalar-curvature singularity. (The local analyses in [22–25] did not reveal the oscillatory character of the singularity, though, because they all *assumed* nonoscillatory local initial data for simplicity [26].)

The CH singularity inside a spinning BH is also found to be essentially linear. Namely, the fully nonlinear curvature singularity at the early portion of the CH is adequately described (at the leading order) by the linear gravitational perturbation over the Kerr background. This is demonstrated in Ref. [20], where it is found that the linear perturbation term dominates all higher-order terms in the nonlinear perturbation expansion. (This holds both in terms of the metric perturbations and in terms of the curvature perturbations derived from them.) This essential linearity is also supported by the recent analysis of plane-symmetric null weak singularities [25] and is consistent with the other local analyses [22–24]. Thanks to this essential linearity (which is an unusual phenomenon in the context of spacetime singularities or singularities in general), one can study the structure and features of the CH singularity, both qualitatively and quantitatively (at the leading order), by a linear analysis. This motivates one to carry out a detailed linear analysis of gravitational perturbations over the Kerr background. Such a linear perturbation analysis is the main objective of this paper.

We preferred here to use the Newman-Penrose formulation for perturbations over the Kerr background (see, e.g., [27]), because it enables the separation of the wave equation [28]. In turn, this separability allows the determination of the leading-order coefficients at the CH [by solving ordinary differential equations (ODEs) in the interval between the event horizon (EH) and the CH]. In Ref. [20] we used the framework of metric perturbations, because it is easier to analyze the nonlinear perturbations in this framework. Because of the nonseparability of metric perturbations over the Kerr background, the ''coefficients'' obtained in Ref. [20] for the asymptotic behavior at the CH were in fact unknown functions of the angular coordinate θ, and it was not even clear which of these coefficients vanish and which do not. Here the separability of the Newman-Penrose fields allows us to carry out the linear part of the analysis in much greater detail and to obtain more complete results. (We note that all the results obtained in this paper are consistent with Ref. [20].)

The present paper is a part of a project aimed at exploring the inner structure of realistic spinning BHs and, particularly, the structure of the null CH singularity. Reference [20] outlines the approach used—a systematic nonlinear perturbation expansion, combined with the late-time expansion. It also describes the main results concerning the structure and features of the CH singularity (i.e., a null, weak, essentially linear, scalar-curvature singularity). The general approach and its relevance to the gravitational collapse of a spinning astrophysical object are extensively discussed in Ref. [29]. The late-time expansion and its implementation for the analysis of perturbations near the CH are demonstrated in detail in Refs. [30,31] for the case of a scalar field in a Reissner-Nordström background. The application of the late-time expansion to an axially symmetric scalar field in a Kerr BH is described in Ref. [32]. The present paper is devoted to the analysis of linear gravitational perturbations inside a Kerr BH, in the Newman-Penrose formalism. In forthcoming papers we shall generalize the analysis to linear metric perturbations and then to nonlinear metric perturbations.

Although the main objective of this paper is the analysis of gravitational perturbations, Teukolsky's master equation [28] for the Newman-Penrose fields allows us to include electromagnetic perturbations in the analysis with almost no modifications. We also include nonaxially symmetric scalar perturbations in this analysis (axially symmetric scalar perturbations require a somewhat different analysis and were already analyzed in Ref. [32]). We shall thus consider in this paper ingoing $(s=2)$ and outgoing $(s=-2)$ gravitational perturbations, and ingoing $(s=1)$ and outgoing $(s=-1)$ electromagnetic perturbations, as well as nonaxially symmetric modes (i.e., $m\neq 0$, where m is the mode's magnetic number) of a scalar field $(s=0)$ [33].

The structure of this paper is similar to that of Ref. [32]. In Sec. II we introduce the field equation, the coordinates, and some additional notation. In Sec. III we describe the initial-value setup for our problem. We consider a characteristic initial-value problem, in which one of the null hypersurfaces is located at the EH. To that end, we need to know the behavior of perturbations along the EH (which in turn requires the analysis of perturbations outside the BH). In Sec. III we describe recent results concerning the late-time behavior of perturbations outside the Kerr BH and, particularly, along the EH.

Section IV describes the late-time expansion for the Newman-Penrose fields. Then, in Sec. V we use this expansion to obtain a general expression for the asymptotic behavior near the CH. This asymptotic behavior includes inverse powers of u or v, whose coefficients are unspecified at this stage. In order to obtain explicit values for these coefficients, we have to match the various parameters to the presumed initial data at the EH. To that end, in Sec. VI we analyze the local asymptotic behavior near the EH, as dictated by the late-time expansion (and by the demand for regularity). Then in Sec. VII we decompose the Newman-Penrose fields and all related functions into spin-weighted spherical harmonics. This decomposition completely separates the field equation for the first term in the late-time expansion, which we denote ψ_0. As a consequence, the field equation for ψ_0 becomes an ordinary differential equation, which we solve explicitly in Sec. VIII. Then in Sec. IX we use this exact solution to obtain the leading-order coefficients at the CH by matching them to the corresponding coefficients at the EH. By this we obtain the explicit asymptotic behavior of the perturbation at the early portion of the CH. Our main results are given in Eq. (111) for $s=1,2$ and in Eq. (115) for $s=-1,-2$. Finally, in Sec. X we summarize the main results and give some concluding remarks. Some details of the calculations are left to the Appendixes.

II. FIELD EQUATION

Let Ψ denote a linear Teukolsky field with integer spin-weight $-2\leq s\leq 2$ (see Ref. [28], and note that our Ψ is denoted ψ there. The relation of Ψ to the standard Newman-Penrose fields in the Kinnersley's null tetrad is given in Table I therein). This includes the gravitational perturbations $(s=\pm 2)$, electromagnetic perturbations $(s=\pm 1)$, or a scalar perturbation $(s=0)$. The background geometry considered here is a nonextreme Kerr BH, with mass M and specific angular momentum a $(0<|a|<M)$. Let (r,t,θ,φ) be the standard Boyer-Lindquist coordinates for the Kerr geometry. We first decompose Ψ into azimuthal modes $e^{im\varphi}$:

$$\Psi(r,t,\theta,\varphi)=\sum_m \psi_m(r,t,\theta)e^{im\varphi}\equiv\sum_m \Psi_m . \tag{1}$$

Because of the axial symmetry of the Kerr background, the field equation does not couple perturbation modes with different m. Therefore, we shall consider here a perturbation mode with a specific m, and for brevity we shall usually omit the index m from ψ_m and Ψ_m. The reader should recall that throughout this paper ψ (or Ψ) always carries two indices: m and s.

The field equation for ψ is obtained from the master equation in Ref. [28] by substituting $\partial_\varphi\to im$:

$$\begin{aligned}&\left[\frac{(r^2+a^2)^2}{\Delta}-a^2\sin^2\theta\right]\psi_{,tt}+\frac{4iamMr}{\Delta}\psi_{,t}\\&\quad-m^2\left[\frac{a^2}{\Delta}-\frac{1}{\sin^2\theta}\right]\psi-\Delta^{-s}\partial_r(\Delta^{s+1}\psi_{,r})\\&\quad-\frac{1}{\sin\theta}\partial_\theta(\sin\theta\psi_{,\theta})-2ims\left[\frac{a(r-M)}{\Delta}+\frac{i\cos\theta}{\sin^2\theta}\right]\psi\\&\quad-2s\left[\frac{M(r^2-a^2)}{\Delta}-r-ia\cos\theta\right]\psi_{,t}\\&\quad+(s^2\cot^2\theta-s)\psi=0,\end{aligned} \tag{2}$$

where $\Delta\equiv r^2-2Mr+a^2$. It is useful to reexpress Δ as

$$\Delta=(r-r_+)(r-r_-), \tag{3}$$

where r_+ and r_- denote the r values of the event horizon and the inner horizon, respectively, which are given by $r_\pm=M\pm\sqrt{M^2-a^2}$.

For the method we use below it is useful to distinguish between t derivatives and the other derivatives in the field equation (2). We therefore rewrite this equation as

$$(D-T_1\partial_t-T_2\partial_{tt})\psi=0, \tag{4}$$

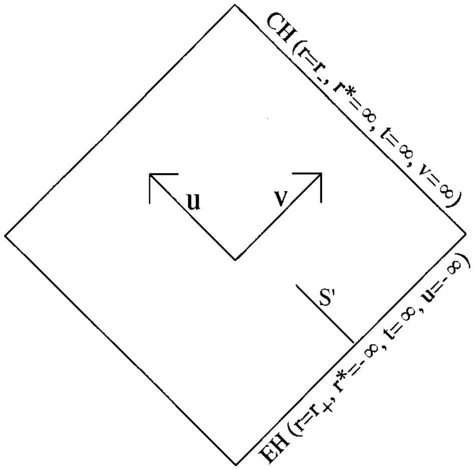

FIG. 2. The region between the two horizons (the region marked II in Fig. 1). This figure shows the range of the Eddington coordinates u and v, and also the values of r, r^*, and t on the two horizons. S' denotes a null hypersurface intersecting the EH, used (along with the EH) in a characteristic initial-value setup for the black hole interior.

where

$$D\equiv\Delta^{1-s}\partial_r(\Delta^{1+s}\partial_r)+a^2m^2+2isam(r-M)+\Delta D_\theta, \tag{5}$$

$$D_\theta=\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta)-\frac{m^2}{\sin^2\theta}-2sm\frac{\cos\theta}{\sin^2\theta}-(s^2\cot^2\theta-s),$$

and the coefficients T_1 and T_2 are given by

$$T_1=4iamMr-2sM(r^2-a^2)+2s(r+ia\cos\theta)\Delta, \tag{6}$$

$$T_2=(r^2+a^2)^2-a^2\sin^2\theta\Delta. \tag{7}$$

For later use, we also define the ''tortoise coordinate'' r^* by

$$dr/dr^*\equiv\Delta/(r^2+a^2)$$

and the Eddington coordinates u,v by

$$u\equiv r^*-t,\quad v\equiv r^*+t.$$

The range of these coordinates inside the BH is shown in Fig. 2.

The surface gravity at the event and the Cauchy horizons is given by

$$\kappa_\pm=\frac{\delta}{2(r_\pm^2+a^2)}=\frac{\delta}{4Mr_\pm},$$

where

$$\delta\equiv r_+-r_-=2\sqrt{M^2-a^2}.$$

III. INITIAL DATA AT THE EVENT HORIZON

In studying the evolution of perturbations inside the BH, it is natural to consider a characteristic initial-value setup in which one of the two initial null hypersurfaces is the EH. The other characteristic null hypersurface is denoted S' in Fig. 2. In this paper we are dealing with the internal late-time behavior, i.e., the behavior at fixed r and large t inside the BH, and the behavior at the early portion of the CH. As was explained in Ref. [29] (see also [30,32]), this late-time behavior is not sensitive to the initial data on the characteristic hypersurface S'. (This follows from the physical demand for the regularity of the initial perturbation, combined with the exponential redshift along the EH.) Therefore, in order to analyze the late-time internal dynamics, it is sufficient to know the value of the perturbation on the EH (particularly at large v). This, in turn, requires the analysis of perturbations *outside* the BH.

For a Schwarzschild background it is well known since Price's work [34] that perturbations outside the BH eventually decay as an inverse power of external time. That is, a perturbation mode of multipolar index l decays at fixed r as t^{-n_0}, with $n_0=2l+3$. (If the mode under consideration initially has a nonvanishing static multipole moment, then the power index is $2l+2$ [34]. However, throughout this paper we shall only consider the situation of vanishing initial static multipole moment.)

In the case of a Kerr background the problem is more complicated due to the lack of spherical symmetry. Preliminary calculations [29,35] suggested a power-law decay which is similar to the Schwarzschild case, except for two important differences: (i) There is a coupling between spherical-harmonics multipoles of different l (but with the same m), and (ii) at the EH the perturbation oscillates in the Eddington coordinate v along the horizon's null generators, with a frequency proportional to m. A numerical simulation of the evolution of Ψ outside a Kerr BH was carried out by Krivan *et al.* for the cases $s=0$ [36] and $s=-2$ [37]. More recently, systematic analytic calculations for a scalar field were carried out by Barack and Ori [38] and by Hod [39]. These analyses revealed that at fixed r and large t, a scalar-field multipole of given l,m generically decays with a power index $l+|m|+3+q$, where q is 0 for even $l+m$ and 1 for odd $l+m$ [40]. At the EH, the scalar field decays as an inverse power of v with the same power index as above, and for $m\neq0$ it also oscillates with v (as described in Ref. [29]).

For $s\neq0$ fields, the late-time behavior outside the spinning BH was analyzed recently by Hod [41] and by Barack [42]. In this case, the power index at fixed r is generically

$$l+l_0+3, \tag{8}$$

where l_0 is the minimal value of l allowed for the values m,s of the multipole under consideration, that is,

$$l_0=\max(|m|,|s|)\quad(s\neq0). \tag{9}$$

[In order to incorporate the case $s=0$ in Eq. (8), we define in this case $l_0=|m|+q$.] There also is an additional factor Δ^{-s} at the EH [see Eq. (12) below; this factor has a trivial origin, as explained in Ref. [28]]. Note that for both $s=0$ and $s\neq 0$, the decay rate of an individual multipole, $l>l_0$, is slower than the Schwarzschild case, due to the coupling to multipoles of smaller l (caused by the breakdown of spherical symmetry) [40].

For the purpose of the present analysis it will be sufficient to consider the *overall* late-time behavior for given *m,s*. This overall behavior is dominated by the multipole $l=l_0$, which has the slowest decay rate. At fixed *r* and large *t*, the overall late-time decay is given by

$$\Psi \propto {}_sY^m_{l_0}(\theta,\varphi)t^{-n_0}+O(t^{-n_0-1}) \quad (\mathrm{EH}), \tag{10}$$

where

$$n_0=2l_0+3, \tag{11}$$

and ${}_sY^m_l$ denotes the spin-weighted spherical harmonics. (Note that the spheroidal harmonics are not useful for the final description of the late-time behavior, because they are defined in the frequency domain and not in the time domain.) The overall inverse-power decay (10), (11) has been demonstrated numerically in [36,37].

Once the late-time decay at fixed *r* is known, the asymptotic behavior at the EH may be derived by purely local considerations [43]. One obtains the following late-time behavior:

$$\Psi=C_0\,{}_sY^m_{l_0}(\theta,\phi_+)\Delta^{-s}e^{im\Omega_+v}v^{-n_0}+O(v^{-n_0-1}) + O(\Delta^{1-s}) \quad (\mathrm{EH}), \tag{12}$$

where $\Omega_+\equiv a/(2Mr_+)$, and ϕ_+ is an azimuthal coordinate regularized at the EH:

$$\phi_+\equiv\varphi-\Omega_+t$$

(the Boyer-Lindquist azimuthal coordinate φ is singular at the EH of a Kerr BH [27]). C_0 is an unspecified coefficient (this coefficient vanishes in certain cases; see below). The asymptotic behavior including the higher-order terms in $1/v$ takes the form [44]

$$\Psi=\Delta^{-s}e^{im\Omega_+v}\sum_l {}_sY^m_l(\theta,\phi_+)\sum_{j=0}^{\infty} C^l_j v^{-n_0-j} + O(\Delta^{1-s}) \quad (\mathrm{EH}). \tag{13}$$

To relate the two representations (12) and (13), note that in the latter the coefficients for the term $j=0$ are given by

$$C^l_{j=0}=C_0\delta_{l,l_0}. \tag{14}$$

It is sometimes useful to express the perturbation field without decomposing it into spin-weighted spherical harmonics. Separating the harmonics ${}_sY^m_l(\theta,\phi_+)$ into an azimuthal part $e^{im\phi_+}$ and a θ-dependent part (and absorbing the latter into the coefficients C^j), Eq. (13) takes the form

$$\Psi=e^{im\phi_+}\Delta^{-s}e^{im\Omega_+v}\sum_{j=0}^{\infty} C^j(\theta)v^{-n_0-j}+O(\Delta^{1-s}) \quad (\mathrm{EH}). \tag{15}$$

In Ref. [43] we show that $C_0\neq 0$ for $s<0$ and/or $am\neq 0$, but C_0 vanishes in the case $s>0$, $am=0$ (i.e., for all $s>0$ fields in the Schwarzschild case and for all axially symmetric $s>0$ modes in the Kerr case) [45]. In this case, the asymptotic behavior at the EH will be governed by the term $j=1$ in Eq. (13) or (15). The calculations throughout this paper are valid in both cases $C_0=0$ and $C_0\neq 0$. (However, the vanishing of C_0 for $s>0$, $m=0$ will have obvious implications to the divergence rate of these perturbation modes at the CH, as we discuss in Sec. IX.)

IV. LATE-TIME EXPANSION

The form (15) of the initial data at the EH suggests that the perturbation field inside the BH (like the field outside the BH) will be of the form

$$\psi(r,t,\theta)=\sum_{k=0}^{\infty} \psi_k(r,\theta)t^{-n-k}, \tag{16}$$

which we refer to as the *late-time expansion*. (This issue is discussed in more detail in Refs. [29,30].) This ansatz is consistent with the field equation and also with the initial data (15) at the EH, as we show below. The constant *n* is an integer greater than 1 [by matching Eq. (16) to the initial data at the EH, we shall later show that $n=n_0$].

Substituting the expansion (16) in the field equation (4), we find that the latter is satisfied, provided that the functions ψ_k satisfy

$$D(\psi_k)=N^1_kT_1\psi_{k-1}+N^2_kT_2\psi_{k-2}\equiv S_k, \tag{17}$$

where

$$N^1_k\equiv 1-n-k, \quad N^2_k\equiv(n+k-2)(n+k-1). \tag{18}$$

Note that $S_{k=0}\equiv 0$, because, by definition, $\psi_{k'}$ vanishes for $k'<0$. Therefore, Eq. (17) is a homogeneous equation for $k=0$ and an inhomogeneous equation for $k\geqslant 1$. It forms a hierarchy of equations, one for each *k*, which in principle can be solved one at a time (first for $k=0$, then for $k=1$, etc.).

For later convenience we also reexpress the source term S_k as

$$S_k=\overline{S_k}+\hat{S}_k+\tilde{S}_k, \tag{19}$$

where

$$\overline{S_k}=N^1_k\overline{T_1}\psi_{k-1}+N^2_k\overline{T_2}\psi_{k-2},$$

$$\hat{S}_k=2isaN^1_k\Delta\cos\theta\psi_{k-1},$$

$$\tilde{S}_k=-a^2N^2_k\Delta\sin^2\theta\psi_{k-2}.$$

Here $\overline{T_{1,2}}$ denote the θ-independent part of $T_{1,2}$, that is,

$$\overline{T_1}=4iamMr-2sM(r^2-a^2)+2sr\Delta$$

and

$$\overline{T_2}=(r^2+a^2)^2.$$

V. ASYMPTOTIC BEHAVIOR NEAR THE CAUCHY HORIZON

We shall now analyze the asymptotic behavior at the CH, to the leading order in Δ. To that end, we define

$$R\equiv e^{-2\kappa_-}.$$

R, like Δ, vanishes at the CH, and in the neighborhood of the latter we have $R\propto r-r_-\propto\Delta$. We now expand the field equation in R near $R=0$ and consider the leading-order term only. It is essential that all the θ-dependent terms (and the θ derivatives) in Eqs. (5)–(7) are proportional to Δ and, hence, do not appear at the leading order. Let us denote the leading-order parts in D, T_1, and T_2 by a tilde:

$$D=\tilde{D}+O(R),\quad T_{1,2}=\tilde{T}_{1,2}+O(R).$$

We shall transform the independent variable in D from r to R. At the CH, $\Delta_{,r}=-\delta$, so $\partial_r\cong-\delta\partial_\Delta$. To the leading order we have[1]

$$\Delta^{1-s}\partial_r(\Delta^{1+s}\partial_r)\cong\delta^2\Delta^{1-s}\partial_\Delta(\Delta^{1+s}\partial_\Delta)$$
$$\cong\delta^2R^{1-s}\partial_R(R^{1+s}\partial_R) \qquad (20)$$

and, also,

$$r-M\cong r_--M=-\delta/2.$$

We thus define

$$\tilde{D}\equiv\delta^2R^{1-s}\partial_R(R^{1+s}\partial_R)+a^2m^2-isam\delta\cong D$$

and

$$\tilde{T}_1\equiv2Mr_-(2iam+s\delta)\cong T_1,$$

$$\tilde{T}_2\equiv(r_-^2+a^2)^2\cong T_2.$$

(Throughout this paper, the symbol ''$\cong$'' denotes equality to the leading order in Δ.) In the derivation of $\tilde{T}_1$ we have used the equality $r_-^2-a^2=-r_-\delta$. The field equation (4) then takes the approximate form

$$(\tilde{D}-\tilde{T}_1\partial_t-\tilde{T}_2\partial_{tt})\psi\cong0. \qquad (21)$$

The analysis is further simplified by defining

$$\hat{D}\equiv\delta^{-2}\tilde{D},\quad \hat{T}_{1,2}\equiv\delta^{-2}\tilde{T}_{1,2} \qquad (22)$$

and $A\equiv am/\delta$. We find

$$\hat{D}=R^{1-s}\partial_R(R^{1+s}\partial_R)+(A^2-isA)$$
$$=R^2\partial_{RR}+(1+s)R\partial_R+(A^2-isA) \qquad (23)$$

and

$$\hat{T}_1=\mu(2iA+s),\quad \hat{T}_2=\mu^2, \qquad (24)$$

where $\mu\equiv(2\kappa_-)^{-1}=2Mr_-/\delta=(r_-^2+a^2)/\delta$. The field equation (21) now reads

$$(\hat{D}-\hat{T}_1\partial_t-\hat{T}_2\partial_{tt})\psi\cong0, \qquad (25)$$

and applying it to the late-time expansion, Eq. (16), we obtain

$$\hat{D}(\psi_k)\cong N_k^1\hat{T}_1\psi_{k-1}+N_k^2\hat{T}_2\psi_{k-2}. \qquad (26)$$

It is useful to *define* $\hat{\psi}_k$ to be the *exact* solution of the system (26), i.e.,

$$\hat{D}(\hat{\psi}_k)=N_k^1\hat{T}_1\hat{\psi}_{k-1}+N_k^2\hat{T}_2\hat{\psi}_{k-2}. \qquad (27)$$

We then define $\hat{\psi}$ via the functions $\hat{\psi}_k$ in analogy with Eq. (16):

$$\hat{\psi}(r,t,\theta)\equiv\sum_{k=0}^{\infty}\hat{\psi}_k(r,\theta)t^{-n-k}. \qquad (28)$$

The field $\hat{\psi}$ thus obtained is the *exact* solution of Eq. (25); that is, it satisfies

$$(\hat{D}-\hat{T}_1\partial_t-\hat{T}_2\partial_{tt})\hat{\psi}=0. \qquad (29)$$

[To verify this equality, just apply this differential operator to the right-hand side of Eq. (28), collect terms of the same power in t, and recall Eq. (27).]

For later convenience we also express the differential operator $\hat{D}$ as

$$\hat{D}=\mu^2\partial_{r*r*}-\mu s\partial_{r*}+A^2-isA. \qquad (30)$$

Denoting the differential operator at the left-hand side of Eq. (29) by $\overline{D}$, we find

$$\overline{D}=\mu^2(\partial_{r*r*}-\partial_{tt})-\mu s(\partial_{r*}+\partial_t)-2iA\mu\partial_t+A^2-isA,$$

and the field equation (29) reads

$$\overline{D}(\hat{\psi})=0. \qquad (31)$$

We turn now to analyze the functions $\hat{\psi}_k$ and $\hat{\psi}$. The former are obtained by solving the individual equations included in the system (27) one by one. We start from $k=0$, for which the source term vanishes and the field equation reduces to

[1]The extra first-order derivative terms which arise in this transformation (e.g., the one proportional to the derivative of $\Delta_{,r}$) are smaller by a factor Δ compared to the dominant first-order term in this differential operator and can therefore be neglected here.

$$\hat{D}(\hat{\psi}_0)=0.$$

The general solution of this equation is

$$\hat{\psi}_0(r,\theta)=e_u(\theta)R^{q_u}+e_v(\theta)R^{q_v}, \tag{32}$$

where

$$q_u=iA, \quad q_v=-iA-s \tag{33}$$

and e_u and e_v are (yet) arbitrary functions of θ. (The reason for associating the indices u, v with these two basis solutions will become apparent at the end of this section.) Equation (32) is indeed the general solution, provided that $q_u \neq q_v$. Since both A and s are real, this inequality is satisfied as long as either $A\neq 0$ or $s\neq 0$ (or both). The special case $s=A=0$ is excluded here: This case, which corresponds to an axially symmetric scalar field, requires a special treatment and was already analyzed in Ref. [32].

For later reference, we note that

$$R^{q_u}=e^{-im\Omega_- r^*}, \quad R^{q_v}=R^{-s}e^{im\Omega_- r^*}\cong(\Delta_R)^s\Delta^{-s}e^{im\Omega_- r^*}, \tag{34}$$

where

$$\Omega_-\equiv a/(2Mr_-)=2a\kappa_-/\delta \tag{35}$$

and Δ_R is a number (which depends on a/M only), defined by

$$\Delta_R\equiv \lim_{r\to r_-}(\Delta/R).$$

For $k>0$, Eq. (27) is inhomogeneous. In Appendix A we show that the general exact solution of this system takes the form

$$\hat{\psi}_k=p_k^u(r^*,\theta)R^{q_u}+p_k^v(r^*,\theta)R^{q_v}, \tag{36}$$

where p_k^u and p_k^v are two polynomials of order k in r^*,

$$p_k^w(r^*,\theta)=\sum_{i=0}^{k} p_{ki}^w(\theta)r^{*i}. \tag{37}$$

Hereafter, w stands for either u or v. Furthermore, these two polynomials have the following properties: (1) For each k, the two coefficients $p_{k,i=0}^w$ are arbitrary functions of θ; (2) for $0<i\leqslant k$, p_{ki}^w is uniquely determined as a linear combination of coefficients $p_{k'i'}^w$, with $k'=k-1$ and $k'=k-2$ (and the same w); (3) the k'th-order polynomial $p_k^w(r^*)$ is nondegenerate, i.e., $p_{k,i=k}^w\neq 0$, if and only if e_w is nonzero. (Furthermore, the coefficients $p_{k,i=k}^w$ are all proportional to e_w.) Note that properties (1),(2) imply that the two polynomials p_k^u and p_k^v are completely independent of each other.

Equation (36) shows that $\hat{\psi}_k$ is made of two components, distinguished from each other by the value of the power index of R. We denote these two components by $\hat{\psi}_k^u$ and $\hat{\psi}_k^v$, that is,

$$\hat{\psi}_k^w=p_k^w(r^*,\theta)R^{q_w}, \quad w=u,v, \tag{38}$$

and Eq. (36) becomes

$$\hat{\psi}_k=\hat{\psi}_k^u+\hat{\psi}_k^v.$$

We also define $\hat{\psi}^w$ to be the contribution of $\hat{\psi}_k^w$ to $\hat{\psi}$ via Eq. (28):

$$\hat{\psi}^w=\sum_{k=0}^{\infty}\hat{\psi}_k^w t^{-n-k}, \quad w=u,v, \tag{39}$$

so

$$\hat{\psi}=\hat{\psi}^u+\hat{\psi}^v. \tag{40}$$

From this point the calculation proceeds in close analogy with that of Ref. [32]. Combining Eqs. (37)–(39), we obtain

$$\hat{\psi}^w=R^{q_w}\sum_{i=0}^{k}\sum_{k=0}^{\infty}p_{ki}^w r^{*i}t^{-n-k}. \tag{41}$$

We define

$$f_j^w\equiv R^{q_w}\sum_{k=j}^{\infty}p_{k,i=k-j}^w r^{*k-j}t^{-n-k} \quad (j\geqslant 0),$$

and find

$$\hat{\psi}^w=\sum_{j=0}^{\infty}f_j^w \tag{42}$$

[that is, the term $j=0$ corresponds to all terms $i=k$ in Eq. (41), the term $j=1$ corresponds to the terms $i=k-1$, etc.]. Next, we define $\hat{w}=r^*/t$ and express f_j^w as

$$f_j^w\equiv R^{q_w}t^{-n-j}F_j^w(\hat{w},\theta), \tag{43}$$

where

$$F_j^w\equiv\sum_{k=j}^{\infty}p_{k,i=k-j}^w\hat{w}^{k-j}.$$

We now define $\overline{w}=u/v$ and rewrite Eq. (43) as

$$f_j^w\equiv R^{q_w}u^{-n-j}\overline{F}_j^w(\overline{w},\theta), \tag{44}$$

where

$$\overline{F}_j^w(\overline{w},\theta)\equiv\left(\frac{2\overline{w}}{1-\overline{w}}\right)^{n+j}F_j^w(\hat{w}(\overline{w}),\theta)$$

[we have used here $t=u(1-\overline{w})/2\overline{w}$ and $\hat{w}=(1+\overline{w})/(1-\overline{w})$].

In the next stage we use the field equation (31) and apply the differential operator $\overline{D}$ to $\hat{\psi}=\hat{\psi}^u+\hat{\psi}^v$. From the mutual independence of $\hat{\psi}_k^u$ and $\hat{\psi}_k^v$ in the above construction [which follows from properties (1),(2) above], it is obvious that

$\overline{D}(\hat{\psi}^w)$ must vanish independently for $w=u$ and $w=v$. [This is also evident from the way the operator $\overline{D}$ acts on $\hat{\psi}^u$ and $\hat{\psi}^v$: $\overline{D}$ does not modify the power index of R (see Appendix B); hence (as $q_u \neq q_v$) it does not mix terms from $\hat{\psi}^u$ with terms from $\hat{\psi}^v$.] Substituting Eqs. (42),(44) in $\overline{D}(\hat{\psi}^w)=0$, we obtain

$$0=\sum_{j=0}^{\infty} \overline{D}(f_j^w)=\sum_{j=0}^{\infty} \overline{D}[R^{q_w}u^{-n-j}\overline{F}_j^w(\overline{w},\theta)]. \tag{45}$$

Consider the contribution of a particular j to the last expression. This contribution includes terms proportional to u^{-n-j-1} and u^{-n-j-2} [each multiplied by R^{q_w} and by some function of $\overline{w}$; note that a term proportional to u^{-n-j} does not appear, because $\overline{D}(R^{q_w})=0$]. One might therefore expect that Eq. (45) will mix terms of different j. In Appendix B, however, we show by a direct calculation that this is not the case, and $\overline{D}(f_j^w)$ vanishes separately for each j. One obtains a simple ordinary differential equation for each function $\overline{F}_j^w$, whose general solution is

$$\overline{F}_j^u=a_j(\theta), \quad \overline{F}_j^v(\overline{w},\theta)=\overline{b}_j(\theta)\overline{w}^{n+j}, \tag{46}$$

where $a_j(\theta)$ and $\tilde{b}_j(\theta)$ are arbitrary functions of θ. A substitution of these results in Eqs. (42) and (44) yields

$$\hat{\psi}^u=\sum_{j=0}^{\infty} f_j^u=R^{iA}\sum_{j=0}^{\infty} a_j(\theta)u^{-n-j} \tag{47}$$

and

$$\hat{\psi}^v=\sum_{j=0}^{\infty} f_j^v=R^{-s-iA}\sum_{j=0}^{\infty} \tilde{b}_j(\theta)v^{-n-j}. \tag{48}$$

Returning from $\hat{\psi}$ to ψ [which amounts to recognizing the presence of $O(\Delta)$ corrections], we split the latter into two parts,

$$\psi=\psi^u+\psi^v,$$

in accordance with Eq. (40). Using Eq. (34), we obtain, from Eqs. (47),(48),

$$\psi^u=e^{-im\Omega_- r^*}\sum_{j=0}^{\infty} a_j(\theta)u^{-n-j}[1+O(\Delta)] \quad \text{(CH)} \tag{49}$$

and

$$\psi^v=\Delta^{-s}e^{im\Omega_- r^*}\sum_{j=0}^{\infty} b_j(\theta)v^{-n-j}[1+O(\Delta)] \quad \text{(CH)}, \tag{50}$$

where $b_j=(\Delta_R)^s\tilde{b}_j$ [we may regard $b_j(\theta)$ as the freely specifiable functions of θ, instead of $\tilde{b}_j(\theta)$].

To interpret these results, and especially the oscillations in r^*, we must keep in our mind that (i) ψ is just the coefficient of the azimuthal mode $e^{im\varphi}$ in the full perturbation field $\Psi(r,t,\theta,\varphi)$, and (ii) the coordinate φ goes singular at the CH of the Kerr geometry. More specifically, φ grows unboundedly along regular world lines which intersect the CH. In order to uncover the physical behavior of the perturbation field at the CH, we shall transform from φ to a new regular azimuthal coordinate

$$\phi\equiv\varphi-\Omega_- t. \tag{51}$$

This new coordinate (together with $U\equiv -e^{-\kappa_- u}$, $V\equiv -e^{-\kappa_- v}$, and θ) regularizes the line element (see, e.g., [27]), and along the world line of an infalling observer, ϕ gets a finite value at the CH. We naturally divide the full perturbation field $\Psi=\psi(r,t,\theta)e^{im\varphi}$ into two components, $\Psi=\Psi^u+\Psi^v$, according to $\Psi^w=\psi^w e^{im\varphi}$. From Eqs. (49),(50) we obtain

$$\Psi^u=e^{im\phi}e^{-im\Omega_- u}\sum_{j=0}^{\infty} a_j(\theta)u^{-n-j}+O(\Delta) \quad \text{(CH)} \tag{52}$$

and

$$\Psi^v=e^{im\phi}\Delta^{-s}e^{im\Omega_- v}\sum_{j=0}^{\infty} b_j(\theta)v^{-n-j}+O(\Delta^{1-s}) \quad \text{(CH)}. \tag{53}$$

Later we shall also need the relations between the leading-order coefficients $a_{j=0},b_{j=0}$ and the leading-order coefficients of the late-time expansion, e_u,e_v. Substituting $u=-t(1-r^*/t)$ in Eq. (47) and $v=t(1+r^*/t)$ in Eq. (48), collecting terms of the same power of $1/t$, and matching the term proportional to t^{-n} to Eq. (32), one finds

$$a_{j=0}=(-1)^n e_u, \quad b_{j=0}=(\Delta_R)^s e_v. \tag{54}$$

(This relation should *not* be interpreted as if the terms $k>0$ do not contribute to the leading-order $(j=0)$ term at the CH. In fact, the terms $k>0$ do contribute to $a_{j=0}$ and $b_{j=0}$, due to the divergence of p_k^w like r^{*k}. However, the coefficients $p_{k,i=k}^w$ of these dominant terms are all proportional to e_u and e_v [cf. property (3) above], which explains the relation (54).)

VI. ASYMPTOTIC BEHAVIOR NEAR THE EVENT HORIZON

In this section we shall analyze the asymptotic behavior at the EH, as dictated by the late-time expansion (16) and the field equation. This will allow us to check the consistency of the late-time expansion with the presumed initial data at the EH. It will also provide the boundary conditions for the functions ψ_k and the value of the parameter n.

The analysis of the asymptotic behavior at the EH proceeds in close analogy with the corresponding analysis at the CH: One simply needs to replace the inner-horizon parameters by the corresponding event-horizon parameters. We define

$$R_+(r)\equiv e^{2\kappa_+ r^*},$$

and note that R_+ vanishes at the EH like $r-r_+$ and Δ. At the EH, $\Delta_{,r}=\delta$, so Eq. (20) is still valid (with R replaced by R_+), but this time $r-M\cong r_+-M=+\delta/2$. Therefore, the operator $\tilde{D}_+$, which denotes the EH analogue of $\tilde{D}$, has the same form as the latter except that δ is replaced by $-\delta$ and R by R_+. Defining $\hat{D}_+\equiv\delta^{-2}\tilde{D}_+$, we obtain

$$\hat{D}_+=R_+^{1-s}\partial_{R_+}(R_+^{1+s}\partial_{R_+})+(A^2+isA). \tag{55}$$

Let us denote the EH analogue of $\tilde{T}_{1,2}$ by $\tilde{T}_{1,2}^+$. Recalling that $r_+^2-a^2=+r_+\delta$, one finds

$$\tilde{T}_1^+\equiv 2Mr_+(2iam-s\delta)\cong T_1,$$

$$\tilde{T}_2^+\equiv(r_+^2+a^2)^2\cong T_2.$$

Defining $\hat{T}_{1,2}^+\equiv\delta^{-2}\tilde{T}_{1,2}^+$, we obtain

$$\hat{T}_1^+=\mu_+[2iA-s],\quad \hat{T}_2^+=\mu_+^2, \tag{56}$$

where

$$\mu_+\equiv(2\kappa_+)^{-1}=2Mr_+/\delta=(r_+^2+a^2)/\delta.$$

Let $\hat{\psi}_+$ and $\hat{\psi}_k^+$ denote the leading order (in Δ) of ψ and ψ_k, respectively, in the neighborhood of the EH. The functions $\hat{\psi}_k^+$ are defined to be the (exact) solutions of the local system

$$\hat{D}_+(\hat{\psi}_k^+)=N_k^1\hat{T}_1^+\hat{\psi}_{k-1}^+ + N_k^2\hat{T}_2^+\hat{\psi}_{k-2}^+, \tag{57}$$

and $\hat{\psi}_+$ is constructed from $\hat{\psi}_k^+$ by the late-time expansion

$$\hat{\psi}_+(r,t,\theta)\equiv\sum_{k=0}^{\infty}\hat{\psi}_k^+(r,\theta)t^{-n-k}. \tag{58}$$

$\hat{\psi}_+$ satisfies the asymptotic field equation

$$(\hat{D}_+-\hat{T}_1^+\partial_t-\hat{T}_2^+\partial_{tt})\hat{\psi}_+=0. \tag{59}$$

[Compare Eqs. (57)–(59) to Eqs. (27)–(29).]

In principle, we could follow the calculation scheme of Sec. V step by step, with the necessary replacement of CH parameters by the corresponding EH parameters. It is possible, however, to proceed in a simpler way and to obtain the asymptotic behavior at the EH directly from that of the CH. We first recall that in Sec. V both $\hat{\psi}_k$ and $\hat{\psi}$ are completely determined from Eqs. (27)–(29). Now, Eqs. (57)–(59) have exactly the same form as Eqs. (27)–(29), except that

$$\hat{D}\to\hat{D}_+,\quad \hat{T}_1\to\hat{T}_1^+,\quad \hat{T}_2\to\hat{T}_2^+. \tag{60}$$

The latter transformation, however, is achieved by

$$A\to -A,\quad \mu\to-\mu_+,\quad R\to R_+ \tag{61}$$

[compare Eqs. (23),(24) with Eqs. (55),(56)]. Therefore, the general solution for $\hat{\psi}_k^+$ and $\hat{\psi}_+$ is simply obtained from that of $\hat{\psi}_k$ and $\hat{\psi}$ through the transformation (61). Thus $\hat{\psi}_0^+$ will take the simple form

$$\hat{\psi}_0^+(r,\theta)=e_u^+(\theta)R_+^{q_u^+}+e_v^+(\theta)R_+^{q_v^+}, \tag{62}$$

where

$$q_u^+=-iA,\quad q_v^+=iA-s \tag{63}$$

and e_u^+ and e_v^+ are (yet) arbitrary functions of θ. For $k>0$ we have an expression analogous to Eq. (36):

$$\hat{\psi}_k^+=p_k^{+u}(r^*,\theta)R_+^{q_u^+}+p_k^{+v}(r^*,\theta)R_+^{q_v^+}, \tag{64}$$

where again p_k^{+u} and p_k^{+v} are two polynomials of order k in r^*,

$$p_k^{+w}(r^*,\theta)=\sum_{i=0}^{k}p_{ki}^{+w}(\theta)r^{*i}\quad (w=u,v). \tag{65}$$

After substituting Eq. (64) into Eq. (58) and converting the solution from t and r^* to u and v, we find that $\hat{\psi}_+=\hat{\psi}_+^u+\hat{\psi}_+^v$, where

$$\hat{\psi}_+^u=R_+^{-iA}\sum_{j=0}^{\infty}a_j^+(\theta)u^{-n-j} \tag{66}$$

and

$$\hat{\psi}_+^v=R_+^{iA-s}\sum_{j=0}^{\infty}\tilde{b}_j^+(\theta)v^{-n-j}, \tag{67}$$

and $a_j^+(\theta)$ and $\tilde{b}_j^+(\theta)$ are new arbitrary functions of θ [this result may be obtained by applying the transformation rule (61) to Eqs. (47),(48)]. The EH analogue of Eq. (34) is

$$R_+^{q_u^+}=e^{-im\Omega_+r^*},$$

$$R_+^{q_v^+}=R_+^{-s}e^{im\Omega_+r^*}\cong(\Delta_{R_+})^s\Delta^{-s}e^{im\Omega_+r^*}, \tag{68}$$

where

$$Q_+\equiv a/(2Mr_+)=2a\kappa_+/\delta$$

and

$$\Delta_{R_+}\equiv\lim_{r\to r_+}(\Delta/R_+).$$

Returning from $\hat{\psi}_+$ to ψ, using Eq. (68), we find that near the EH ψ is the sum of the two components

$$\psi_+^u = e^{-im\Omega_+ r^*}\sum_{j=0}^{\infty} a_j^+(\theta) u^{-n-j}[1+O(\Delta)] \quad \text{(EH)} \tag{69}$$

and

$$\psi_+^v = \Delta^{-s} e^{im\Omega_+ r^*}\sum_{j=0}^{\infty} b_j^+(\theta) v^{-n-j}[1+O(\Delta)] \quad \text{(EH)} \tag{70}$$

[in analogy with Eqs. (49),(50)], where $b_j^+ = (\Delta_{R_+})^s \tilde{b}_j^+$. Finally, returning from ψ to the full perturbation field $\Psi = \psi(r,t,\theta)e^{im\varphi}$, we conclude that near the EH Ψ is made of two pieces,

$$\Psi = \Psi_+^u + \Psi_+^v,$$

where

$$\Psi_+^u = \psi_+^u e^{im\varphi} = e^{im\phi_+} e^{-im\Omega_+ u}\sum_{j=0}^{\infty} a_j^+(\theta) u^{-n-j} + O(\Delta) \quad \text{(EH)} \tag{71}$$

and

$$\Psi_+^v = \psi_+^v e^{im\varphi} = e^{im\phi_+}\Delta^{-s} e^{im\Omega_+ v}\sum_{j=0}^{\infty} b_j^+(\theta) v^{-n-j} + O(\Delta^{1-s}) \quad \text{(EH)}. \tag{72}$$

$\phi_+ \equiv \varphi - \Omega_+ t$ is the above-mentioned azimuthal coordinate which is regular at the EH.

The analysis of the asymptotic behavior at the EH up to this point was solely based on the local field equations and the late-time expansion. We have not explicitly taken into account yet the presumed initial data (except that these initial data were the motivation for the late-time expansion). Note the consistency of Eq. (72) with Eq. (15). Matching these two equations yields

$$n = n_0 \tag{73}$$

and

$$b_j^+(\theta) = C^j(\theta). \tag{74}$$

In addition, for $s \leqslant 0$ this matching implies that Ψ_+^u is actually absent; that is, the coefficients $a_j^+(\theta)$ must vanish for each j:[2]

$$a_j^+(\theta) = 0 \quad (s \leqslant 0). \tag{75}$$

This equality can also be established from the demand for regularity. Consider the contribution of a nonvanishing coefficient a_j^+ to the components F_{aU_+} or $C_{aU_+bU_+}$ of the Maxwell or Weyl tensors, respectively. Here a,b stand for the (regularized) angular coordinates θ,ϕ_+, and U_+ is the Kruskal-like coordinate $U_+ \equiv e^{\kappa_+ u}$, which is regular at the EH (where $U_+ = 0$). One can easily verify that for $s<0$ this contribution diverges at the EH like $U_+^{-|s|}$ (modulated by oscillations and by some logarithmic factor).

Note also the EH analogue of Eq. (54), that is,

$$a_{j=0}^+ = (-1)^n e_u^+, \quad b_{j=0}^+ = (\Delta_{R_+})^s e_v^+. \tag{76}$$

[For $s<0$, however, Eq. (75) implies that the first equality reduces to $a_{j=0}^+ = e_u^+ = 0$.]

VII. DECOMPOSITION INTO SPHERICAL HARMONICS

We now decompose $\psi(r,t,\theta)$ in spin-weighted spherical harmonics:

$$\psi(r,t,\theta) = \sum_l {}_s\hat{Y}_l^m(\theta)\psi^l(r,t),$$
$$\psi_k(r,\theta) = \sum_l {}_s\hat{Y}_l^m(\theta)\psi_k^l(r), \tag{77}$$

where ${}_s\hat{Y}_l^m(\theta)$ is the θ-dependent part of the standard spin-weighted spherical harmonic ${}_sY_l^m(\theta,\varphi)$, that is, ${}_sY_l^m(\theta,\varphi) = {}_s\hat{Y}_l^m(\theta)e^{im\varphi}$. (Note that due to the completeness of the family of spin-weighted spherical harmonics for each s, this decomposition is well defined regardless of separability.) The late-time expansion (16) now takes the form

$$\psi^l(r,t) = \sum_{k=0}^{\infty} \psi_k^l(r) t^{-n-k}. \tag{78}$$

The functions ψ_k^l satisfy the ordinary differential equation

$$D^l(\psi_k^l) = S_k^l, \tag{79}$$

where

[2]Equation (75) does not necessarily hold for $s>0$, because in this case the contribution from nonvanishing a_j^+ may be overshadowed by the $O(\Delta^{1-s})$ terms in Eq. (15) or Eq. (72). This cannot happen for $s \leqslant 0$, because in this case the contribution from nonvanishing a_j^+ would dominate Ψ (or $\Psi_{,U_+}$ for $s=0$) at the EH. The regularity argument below is also inconclusive in the case $s>0$. [For $s=0$, one may derive Eq. (75) by demanding regularity of the *gradient* of the scalar field.] Note, however, that Eq. (74) is valid for any s: In the case $s>0$, the entire contribution from Eq. (71) to Ψ is negligible; in the case $s<0$, this contribution vanishes because all coefficients a_j^+ vanish. [In the latter case, an investigation of the terms $O(\Delta)$ in Eq. (71), which is beyond the scope of the present paper, yields that these higher-order terms vanish if all coefficients a_j^+ vanish.]

$$D^l \equiv \Delta^{1-s}\partial_r(\Delta^{1+s}\partial_r) + a^2m^2 + 2isam(r-M) - \Delta(l-s)(l+s+1). \tag{80}$$

The source term S_k^l in Eq. (79) is given by

$$S_k^l = \overline{S_k^l} + \hat{S}_k^l + \tilde{S}_k^l, \tag{81}$$

where

$$\overline{S_k^l} = N_k^1 \overline{T_1} \psi_{k-1}^l + N_k^2 \overline{T_2} \psi_{k-2}^l,$$

$$\hat{S}_k^l = 2isaN_k^1 \Delta \sum_{l'} \lambda_{ll'} \psi_{k-1}^{l'},$$

$$\tilde{S}_k^l = -a^2 N_k^2 \Delta \sum_{l'} \eta_{ll'} \psi_{k-2}^{l'}. \tag{82}$$

The parameters $\lambda_{ll'}$ and $\eta_{ll'}$ (which also depend on m and s) are the expansion coefficients of $\cos\theta_s Y_{l'}^m(\theta)$ and $\sin^2\theta_s Y_{l'}^m(\theta)$, respectively, in ${}_sY_l^m(\theta)$. The explicit values of these coefficients are given in Ref. [46]. For our purpose it is sufficient to recall that, for a given l, there are (at most) three nonvanishing coefficients $\lambda_{ll'}$, $l'=l,l\pm1$, and five nonvanishing coefficients $\eta_{ll'}$, $l'=l,l\pm1,l\pm2$.

The source term S_k^l couples modes of different l—a consequence of the lack of spherical symmetry in the Kerr background. However, a function ψ_k^l is only coupled to (finite number of) functions $\psi_{k'}^{l'}$ with $k'<k$. The equations enfolded in Eq. (79) can therefore be solved one at a time.

The asymptotic behavior of ψ^l and ψ_k^l at the two horizons is derived from the expressions in the last two sections by a straightforward decomposition into spin-weighted spherical harmonics. Here we mention those expressions which are important for the analysis below. Equations (32) and (62) become

$$\hat{\psi}_{k=0}^l(r) = e_u^l R^{q_u} + e_v^l R^{q_v} \quad \text{(CH)} \tag{83}$$

and

$$\hat{\psi}_{k=0}^{+l}(r) = e_u^{+l} R_+^{q_u^+} + e_v^{+l} R_+^{q_v^+} \quad \text{(EH)}. \tag{84}$$

Equations (52),(53) and (71),(72) may now be expressed as

$$\Psi^u = e^{-im\Omega_-\mu} \sum_l {}_sY_l^m(\theta,\phi) \sum_{j=0}^{\infty} a_j^l u^{-n_0-j} + O(\Delta) \quad \text{(CH)}, \tag{85}$$

$$\Psi^v = \Delta^{-s} e^{im\Omega_- v} \sum_l {}_sY_l^m(\theta,\phi) \sum_{j=0}^{\infty} b_j^l v^{-n_0-j} + O(\Delta^{1-s}) \quad \text{(CH)} \tag{86}$$

and

$$\Psi_+^u = e^{-im\Omega_+ u} \sum_l {}_sY_l^m(\theta,\phi_+) \sum_{j=0}^{\infty} a_j^{+l} u^{-n_0-j} + O(\Delta) \quad \text{(EH)}, \tag{87}$$

$$\Psi_+^v = \Delta^{-s} e^{im\Omega_+ v} \sum_l {}_sY_l^m(\theta,\phi_+) \sum_{j=0}^{\infty} b_j^{+1} v^{-n_0-j} + O(\Delta^{1-s}) \quad \text{(EH)}. \tag{88}$$

Note also the mode decomposition of Eqs. (54) and (76), which yields

$$a_{j=0}^l = (-1)^{n_0} e_u^l, \quad b_{j=0}^l = (\Delta_R)^s e_v^l \tag{89}$$

and

$$a_{j=0}^{+l} = (-1)^{n_0} e_u^{+l}, \quad b_{j=0}^{+l} = (\Delta_{R_+})^s e_v^{+l}. \tag{90}$$

Also, Eqs. (74) and (75) now become

$$a_j^{+l} = 0 \quad (s \leqslant 0) \tag{91}$$

and

$$b_j^{+l} = C_j^l. \tag{92}$$

In particular, Eq. (14) implies

$$b_{j=0}^{+l} = C_{j=0}^l = C_0 \delta_{l,l_0}. \tag{93}$$

VIII. GLOBAL SOLUTION FOR $k=0$

The asymptotic behavior at the CH is at the leading order governed by the $k=0$ term in the late-time expansion, which dictates the leading-order $(j=0)$ coefficients [cf. Eq. (54) or its mode decomposition (89)]. The functions $\psi_{k=0}^l$ satisfy the homogeneous equation

$$D^l(\psi_0^l) = 0. \tag{94}$$

This is just the field equation of a stationary (i.e., $\omega=0$) perturbation mode *lms* in the Kerr geometry. We express the general solution of this equation as

$$\psi_{k=0}^l = A^l F_u^l(r) + B^l F_v^l(r), \tag{95}$$

where F_u^l and F_v^l are two independent basis solutions, and A^l, B^l are arbitrary constants (to be determined from the initial data at the EH). Occasionally we shall omit the index l from $F_{u,v}^l$ for brevity.

For modes $m \neq 0$, we take the two basis solutions to be

$$F_u^l(r) = \left(\frac{1-x}{1+x}\right)^{-iA} F(s-l, s+l+1; s+1+2iA; (x+1)/2) \tag{96}$$

and [47]

$$F_v^l(r)=\Delta^{-s}\left(\frac{1-x}{1+x}\right)^{iA}\times F(-s-l,-s+1;-s+1-2iA;(x+1)/2). \tag{97}$$

Here,

$$x\equiv\frac{2r-r_+-r_-}{r_+-r_-}, \tag{98}$$

and F denotes the hypergeometric function [48]. Note that both hypergeometric functions are polynomials in our case, because the first index is a nonpositive integer.

In the case $m=0$, A vanishes, and only one of the above two hypergeometric functions is well defined (the one in which the third index is strictly positive). Let us denote this well-defined solution by F_a: F_a is F_u for $s>0$ and F_v for $s<0$. (The case $s=0$, $m=0$ is not considered here: It was already treated in Ref. [32].) The other solution, which we denote F_b, is obtained by the Wronskian method. The Wronskian W of the differential equation (94) may be easily calculated,

$$W=\Delta^{-s-1}\times\text{const}, \tag{99}$$

and F_b is given by

$$F_b(r)=F_a(r)\int^r F_a^{-2}(r')W(r')dr'. \tag{100}$$

This integral is elementary for integer l,s. Note that F_b is well defined and smooth throughout $r_-<r<r_+$: The zeros of F_a (which are all simple) do not cause a singularity in Eq. (100), because of the factor F_a which multiplies the integral.

Later we shall also need the asymptotic form of the two basic solutions at the two horizons, located at $x=1$ (the EH) and $x=-1$ (the CH). Consider first the case $m\neq0$. Both hypergeometric functions have finite nonvanishing values at the two points $x=\pm1$.[3] At the EH,

$$1-x\propto r-r_+\propto\Delta\propto R_+\quad(\text{EH}), \tag{101}$$

and hence

$$F_u(r)\propto R_+^{-iA},\quad F_v(r)\propto R_+^{iA-s}\quad(\text{EH}). \tag{102}$$

At the CH,

$$1+x\propto r-r_-\propto\Delta\propto R\quad(\text{CH}), \tag{103}$$

and so

$$F_u(r)\propto R^{iA},\quad F_v(r)\propto R^{-iA-s}\quad(\text{CH}). \tag{104}$$

Consider next the asymptotic behavior at the two horizons in the case $m=0$. For $s>0$, $F_a=F_u\propto\Delta^0$ (recall $A=0$), and the integrand in Eq. (100) is proportional to Δ^{-s-1}, leading to $F_b\propto\Delta^{-s}$. Defining $F_v\equiv F_b$ in this case ($m=0$, $s>0$), we recover Eqs. (102),(104). For $s<0$, $F_a=F_v\propto\Delta^{-s}$ and the integrand in Eq. (100) is proportional to Δ^{s-1}, leading to $F_b\propto\Delta^0$. Defining $F_u\equiv F_b$ in this case ($m=0$, $s<0$), we again recover Eqs. (102),(104).

The global $k=0$ solution allows us to relate the leading-order coefficients at the EH to those at the CH. The matching of Eqs. (95), (102), and (104) to Eqs. (83) and (84) yields

$$e_u^{+l}\propto A^l\quad(s\leqslant0),\quad e_v^{+l}\propto B^l\quad(s\geqslant0) \tag{105}$$

at the EH, and

$$e_u^{l}\propto A^l\quad(s\leqslant0),\quad e_v^{l}\propto B^l\quad(s\geqslant0) \tag{106}$$

at the CH. [The restriction to $s\geqslant0$ or $s\leqslant0$ results from possible $O(\Delta)$ terms involved in the transition from $\hat\psi$ (or $\hat\psi_+$) to ψ. These $O(\Delta)$ terms may affect the matching of the subdominant components of $\hat\psi$ and $\hat\psi_+$. But the matching of the dominant components, i.e., $\hat\psi^u,\hat\psi_+^u$ for $s\leqslant0$ and $\hat\psi^v,\hat\psi_+^v$ for $s\geqslant0$, is not affected. In Appendix D we shall show, however, that the second proportionality relation in Eq. (105) holds for $s<0$ too.] Consequently,

$$e_u^{l}\propto e_u^{+l}\quad(s\leqslant0),\quad e_v^{l}\propto e_v^{+l}\quad(s\geqslant0). \tag{107}$$

When combined with Eqs. (89),(90), this implies

$$a_{j=0}^{l}\propto a_{j=0}^{+l}\quad(s\leqslant0),\quad b_{j=0}^{l}\propto b_{j=0}^{+l}\quad(s\geqslant0). \tag{108}$$

[The proportionality constants involved in Eqs. (105)–(108) are all trivial nonvanishing numbers whose explicit values are not important for the present discussion.]

IX. ASYMPTOTIC BEHAVIOR AT THE CH: LEADING-ORDER COEFFICIENTS

For the completion of the analysis of the asymptotic behavior at the CH we still need to calculate the leading-order ($j=0$) coefficients in Eq. (85) or (86). It will be convenient to distinguish at this stage between the cases $s>0$, $s<0$, and $s=0$.

A. Cases $s=1$ and $s=2$

In this case, the asymptotic behavior at the two horizons is dominated by Ψ_+^v and Ψ^v, which both diverge like Δ^{-s}, so we need to determine the parameters $b_{j=0}^l$. Combining Eqs. (93) and (108), we find $b_{j=0}^l\propto C_0\delta_{l,l_0}$. It is convenient to rewrite this relation as

$$b_{j=0}^l=C_0\alpha_v\delta_{l,l_0}, \tag{109}$$

[3] The regularity at $x=-1$ and $x=1$ is guaranteed because both hypergeometric functions are polynomials (as the first index is a nonpositive integer). At $x=-1$ the argument vanishes and $F(a,b;c;0)\equiv1$. At $x=1$ the hypergeometric function gets the value

$$F(a,b,c;1)=\Gamma(c)\Gamma(c-a-b)/\Gamma(c-a)\Gamma(c-b).$$

Since a and b are real, but c is not (as $m\neq0$), all four Γ functions are finite and nonvanishing, so the hypergeometric function is nonvanishing.

where α_v is a nonvanishing constant. In Appendix C we calculate α_v explicitly and show that

$$|\alpha_v|=1. \tag{110}$$

Substituting Eq. (109) into Eq. (86), we obtain the overall leading-order asymptotic behavior at the CH:

$$\Psi=C_0\alpha_v\,{}_sY^m_{l_0}(\theta,\phi)\Delta^{-s}e^{im\Omega_-v}v^{-n_0}+O(v^{-n_0-1})$$
$$+O(\Delta^{1-s})\quad (\mathrm{CH},\;s>0). \tag{111}$$

Note that the most dominant m modes at late time (for a given s) are those with $-|s|\leqslant m\leqslant|s|$, which have the minimal value of l_0 (and n_0):

$$l_0=|s|,\quad n_0=2|s|+3. \tag{112}$$

As we pointed out in Sec. III, the coefficient C_0 vanishes for $m=0$, $s>0$ [43]. The term $j=0$ in Eq. (86) or (53) entirely vanishes in this case, and the asymptotic behavior at the CH is dominated by the term $k=1$:

$$\Psi=b_1(\theta)\Delta^{-s}v^{-n_0-1}+O(v^{-n_0-2})$$
$$+O(\Delta^{1-s})\quad(\mathrm{CH},\;s>0,\;m=0). \tag{111$'$}$$

The present evidence from preliminary calculations is that $b_1(\theta)$ has a nonvanishing contribution from $l=l_0$ only [that is, $b_1(\theta)=\mathrm{const}\times{}_sY^0_{l_0}(\theta)$], but this is still to be verified.

B. Cases $s=-1$ and $s=-2$

In this case the perturbation at the CH is dominated by Ψ^u (as Ψ^v decays like $\Delta^{|s|}$), so it will be determined from the parameters a^l_j in Eq. (85). Consider first the (would-be) leading-order term $j=0$. From Eqs. (91) and (108) we find that

$$a^l_{j=0}=0, \tag{113}$$

so the $j=0$ term entirely vanishes. The overall asymptotic behavior for $s<0$ will therefore be determined by the term $j=1$ in Eq. (85)—provided that this term is nonvanishing (at least for one l). The calculation of the coefficients $a^l_{j=1}$ requires the analysis of the $k=1$ term in the late-time expansion. This analysis is carried out in Appendix D, and it yields

$$a^l_{j=1}=C_0\alpha_u\delta_{l,l_0}, \tag{114}$$

where α_u is a constant. Therefore, the leading-order asymptotic behavior at the CH is

$$\Psi=C_0\alpha_u\,{}_sY^m_{l_0}(\theta,\phi)e^{-im\Omega_-u}u^{-n_0-1}+O(u^{-n_0-2})$$
$$+O(\Delta)\quad(\mathrm{CH},\;s<0). \tag{115}$$

In Appendix D we show that at least for the dominant m modes, $-|s|\leqslant m\leqslant|s|$, the parameter α_u is nonvanishing (except perhaps for isolated values of a/M).

C. Case $s=0$

Since this case has not much physical motivation, we shall not present here the detailed calculation of the coefficients, but just quote the final result:

$$\Psi=C_0Y^m_{l_0}(\theta,\phi)[\alpha_ve^{im\Omega_-v}v^{-n_0}+\alpha_ue^{-im\Omega_-u}u^{-n_0-1}]$$
$$+O(v^{-n_0-1})+O(u^{-n_0-2})+O(\Delta)$$
$$(\mathrm{CH},\;s=0,\;m\neq0),$$

where C_0 is the mode's initial amplitude at the EH.

For completeness we also quote here the analogous expression for axially symmetric scalar-field modes, derived in Ref. [32]:

$$\Psi\cong C_0P_{l_0}(\cos\theta)(-1)^{l_0}[Bv^{-n_0}+(-1)^{n_0}Au^{-n_0}]$$
$$+O(v^{-n_0-1})+O(u^{-n_0-1})+O(\Delta)$$
$$(\mathrm{CH},\;s=0,\;m=0),$$

where

$$A=\frac{1}{2}(1-r_+/r_-),\quad B=\frac{1}{2}(1+r_+/r_-).$$

X. SUMMARY

In this paper we have derived explicit expressions for the asymptotic behavior of the Newman-Penrose gravitational and electromagnetic perturbations near the CH of a Kerr BH. In general, the asymptotic expressions include three important ingredients.

(i) Δ factors: Δ^{-s} for $s>0$ and Δ^0 for $s<0$.

(ii) Inverse powers: v^{-n_0} (v^{-n_0-1} for vanishing m) for $s>0$ and u^{-n_0-1} for $s<0$, where generically $n_0=2|s|+3$.

(iii) Oscillations: A factor $e^{im\Omega_-v}$ for $s>0$ and a factor $e^{-im\Omega_-u}$ for $s<0$.

The Δ factors are of primary importance, because they mark the divergence (or otherwise) of the various fields at the CH. One can easily verify that the factor Δ^{-s} for $s>0$ indeed implies a divergence of the Maxwell components F_{aV} and the Weyl components W_{aVbV} like Δ^{-s} at the CH (this divergence is slightly softened by the inverse power of v). Similarly, the factor Δ^0 for $s<0$ implies finite components F_{aU} and W_{aUbU}. Here the indices a and b stand for the two angular coordinates θ and ϕ, and V and U are the two Kruskal-like coordinates of the inner horizon, $V\equiv-e^{-\kappa_-v}$, $U\equiv-e^{-\kappa_-u}$. (The set of coordinates U,V,θ,ϕ is regular at the CH, so the above tensorial components F_{aV} and W_{aVbV} properly express the divergence of the Maxwell and Weyl tensors, respectively.)

The divergence of the gravitational perturbation field $s=2$ indicates the presence of a curvature singularity, where certain components of the Weyl tensor, as measured by a freely falling observer heading towards the CH, diverge. The divergence rate of curvature, expressed in terms of the observer's proper time τ (with $\tau=0$ at the CH), is proportional to

$$\tau^{-2}(\ln|\tau|)^{-n_0}e^{-im(2a/\delta)\ln|\tau|}$$

in the generic nonaxially symmetric case and to $\tau^{-2}(\ln|\tau|)^{-n_0-1}$ for axially symmetric perturbations. It is possible to show that the curvature scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ also diverges at the CH, but this is beyond the scope of the present paper.

For all modes $m\neq 0$, the divergence of the fields $s=1,2$ is modulated by the factor $e^{im\Omega_{-}v}$ which exhibits infinite number of oscillations on the approach to the CH. Note that near the CH the nonoscillatory, axially symmetric $(m=0)$ modes of the fields $s=1,2$ are smaller by (at least) a factor v compared to the corresponding oscillatory modes with $m\neq 0$.

It should be emphasized that although the analysis in this paper was restricted to linear perturbations, it properly describes (at the leading order) the fully nonlinear structure of the curvature singularity at the early portion of the CH. This is because the CH singularity is essentially linear, that is, the nonlinear corrections (e.g., in terms of curvature) are negligible compared to the linear terms analyzed here. This feature (discussed in Sec. I) is deduced by comparing the linear and nonlinear terms in the systematic nonlinear perturbation expansion [20]. For example, by examining the second-order metric perturbation, one can show that the $s=2$ Newman-Penrose field associated with the second-order perturbation is smaller than the corresponding first-order term (111) by (at least) a factor v^{-5} or u^{-5}. In this sense, the analysis presented here and that of Ref. [20] complement each other: The analysis in Ref. [20] ensures that the nonlinear perturbations are negligible, and the analysis here gives much more detailed information about the linear perturbation and, hence, about the overall structure of the CH singularity.

ACKNOWLEDGMENTS

I would like to thank Leor Barack and Patrick Brady for interesting discussions and helpful comments. This research was supported in part by the United States-Israel Binational Science Foundation.

APPENDIX A

We shall seek solutions to the system (27) of the form

$$\hat{\psi}_k=p_k^w(r^*)R^{q_w}\equiv\hat{\psi}_k^w \tag{A1}$$

where $w=u,v$, and $p_k^w(r^*)$ are polynomials of order k in r^* (throughout most of this appendix we omit the argument θ, because the parametric dependence on θ is irrelevant to the analysis here). Applying the operator $\hat{D}$, Eq. (30), to Eq. (A1), and recalling that $\hat{D}(R^{q_w})=0$, we obtain

$$\hat{D}(\hat{\psi}_k^w)=[(p_k^w)''-(s+2q_w)(p_k^w)']R^{q_w}, \tag{A2}$$

where in this appendix a prime denotes ∂_x, and $x\equiv r^*/\mu$. The differential equation (27) now reduces to

$$(p_k^w)''-(s+2q_w)(p_k^w)'=N_k^1\hat{T}_1p_{k-1}^w+N_k^2\hat{T}_2p_{k-2}^w. \tag{A3}$$

We define

$$G_k\equiv(p_k^w)', \quad z\equiv-(s+2q_w), \quad Z_k\equiv\hat{N}_kp_{k-1}^w+\tilde{N}_kp_{k-2}^w \tag{A4}$$

(with $p_{k<0}^w\equiv 0$, and hence $Z_{k=0}=0$), where $\hat{N}_k\equiv N_k^1\hat{T}_1$ and $\tilde{N}_k\equiv N_k^2\hat{T}_2$. The parameter z is nonvanishing, as $z=\pm(s+2iA)\neq 0$ (with a negative sign for $w=u$ and a positive sign for $w=v$). $\hat{N}_k$ is nonvanishing too, as both N_k^1 and $\hat{T}_1$ are nonzero. Equation (A3) now reads

$$G_k'+zG_k=Z_k. \tag{A5}$$

We need to show that this system has a family of polynomial solutions, with a free parameter C_k^w (in fact, a free function of θ) for each k. [Although the general solution of the second-order equation (A3) has two free parameters for each k, it will be sufficient for us to construct a family of polynomial solutions with one free parameter for each k, as we explain below.] More specifically, we shall construct a family of solutions in which p_k^w is a polynomial of order k (and G_k is a polynomial of order $k-1$) in x.

First, we construct the polynomial solution p_k^w explicitly for $k=0$ and $k=1$. For $k=0$, Z_k vanishes, so Eq. (A5) admits the trivial solution $G_0=0$, and therefore $p_{k=0}^w=\text{const}\equiv C_0^w$ (a zero-order polynomial). For $k=1$, Z_k is a constant, $Z_1=\hat{N}_1C_0^w$, so Eq. (A5) has the trivial solution

$$G_1=\hat{N}_1C_0^w/z=\text{const}$$

and, correspondingly,

$$p_{k=1}^w=(\hat{N}_1C_0^w/z)x+C_1^w.$$

We shall now construct the polynomial solution p_k^w for all $k>1$ by induction. Assume that we have already constructed the polynomials p_k^w (with the above properties) for all $k\leqslant\bar{k}-1$, for some $\bar{k}\geqslant 2$. In particular, $p_{\bar{k}-1}^w$ and $p_{\bar{k}-2}^w$ are polynomials of orders $\bar{k}-1$ and $\bar{k}-2$, respectively. Then, $Z_{\bar{k}}$ is a polynomial of order $\bar{k}-1$, which we express as

$$Z_{\bar{k}}=\sum_{i=0}^{\bar{k}-1}Z_{\bar{k}}^ix^i.$$

(Note that from this point and up to the end of this appendix, i is an index of summation and *not* $\sqrt{-1}$.) Let us write Eq. (A5) for $k=\bar{k}$ in the form

$$G_{\bar{k}}=z^{-1}(Z_{\bar{k}}-G_{\bar{k}}').$$

This equation has a unique polynomial solution

$$G_{\bar{k}}=\sum_{i=0}^{\bar{k}-1}G_{\bar{k}}^ix^i, \tag{A6}$$

where $G_{\bar{k}}^{i=\bar{k}-1}=z^{-1}Z_{\bar{k}}^{i=\bar{k}-1}$, and

$$G_{\bar{k}}^i=z^{-1}[Z_{\bar{k}}^i-(i+1)G_{\bar{k}}^{i+1}] \tag{A7}$$

for $0\leqslant i\leqslant \bar{k}-2$. This formula allows us to construct the coefficients $G_{\bar{k}}^{i}$ one by one, i.e., first $G_{\bar{k}}^{i=\bar{k}-2}$, then $G_{\bar{k}}^{i=\bar{k}-3}$, etc., until the entire polynomial $G_{\bar{k}}^{-}$ in Eq. (A6) is constructed.

Transforming back from G_k to the original unknowns p_k^w, we obtain

$$p_{\bar{k}}^{w}=\sum_{i=0}^{\bar{k}} P_{\bar{k}}^{i}x^{i},$$

where $P_{\bar{k}}^{i=0}\equiv C_{\bar{k}}^{w}$ is an arbitrary constant of integration, and

$$P_{\bar{k}}^{i}=i^{-1}G_{\bar{k}}^{i-1}$$

for $0<i\leqslant\bar{k}$. This completes our construction.

We have constructed a family of polynomials of order k in x (or r^*),

$$p_k^w(x,\theta)=\sum_{i=0}^{k} P_k^i(\theta)x^i, \tag{A8}$$

which, when substituted in Eq. (A1), solve the system (27). Transforming back from x to $r^*=\mu x$, we recover Eq. (37), with $p_{ki}^w=\mu^{-i}P_k^i$.

Although the general solution of the second-order differential equation (A3) is obviously a two-parameter family (for each k and w), here the restriction to the one-parameter family of polynomial solutions does not cause any loss of generality, for the following reason. Our goal is to construct the general solution of Eq. (27), which is a two-parameter family for each k. Substituting our polynomial solutions into Eq. (A1) and in $\hat{\psi}_k=\hat{\psi}_k^u+\hat{\psi}_k^v$, we find that overall our general solution indeed depends on two arbitrary parameters for each k, C_k^u and C_k^v, as required.[4]

The above construction of the polynomial solution (A8) guarantees that property (1) is satisfied. Property (2) follows from the form of Z_k in Eq. (A4) [via Eq. (A7)]. To verify property (3), we focus our attention on the highest-order coefficient in Eq. (A8), i.e., $P_k^{i=k}$. The above construction implies that, for $k\geqslant 1$,

$$P_k^{i=k}=(kz)^{-1}\hat{N}_k P_{k-1}^{i=k-1},$$

which (as $\hat{N}_k\neq 0$, and $e_w\equiv p_{00}^w=P_0^0$) leads to property (3).

[4]One may be concerned about the meaning of the other solutions, not included in the one-parameter family constructed here. These are nonpolynomial solutions, which simply transfer the solution from a u type to a v type (i.e., from $\hat{\psi}_k^u$ to $\hat{\psi}_k^v$), and vice versa. Namely, they are proportional to $R^{\pm(q_u-q_v)}$. These solutions are already represented in the general solution of Eq. (27) as polynomial solutions of the other type.

APPENDIX B

By virtue of Eqs. (42),(44), the equation $\bar{D}(\hat{\psi}^w)=0$ implies

$$\sum_{j=0}^{\infty}\bar{D}(f_j^w)\equiv\sum_{j=0}^{\infty}\bar{D}[R^{q_w}u^{-n-j}\bar{F}_j^w(\bar{w},\theta)]=0. \tag{B1}$$

We shall now show that in both cases $w=u$ and $w=v$, $\bar{D}(f_j^w)$ vanishes for each j separately. Then we shall solve this equation and find $\bar{F}_j^w$ for each j.

1. Case $w=u$

A straightforward calculation shows that the contribution of a particular j to Eq. (B1) is

$$\begin{aligned}\bar{D}(f_j^u)&=\bar{D}[R^{iA}u^{-n-j}\bar{F}_j^u(\bar{w},\theta)]\\&=2\mu\bar{w}^2R^{iA}\{[(2iA+s)\bar{F}_{j,\bar{w}}^u]u^{-n-j-1}\\&\quad+2\mu[-\bar{w}\bar{F}_{j,\bar{w}\bar{w}}^u+(n+j-1)\bar{F}_{j,\bar{w}}^u]u^{-n-j-2}\}.\end{aligned} \tag{B2}$$

For $j=0$ this expression includes a term proportional to u^{-n-1}. Since no other j contributes a term with the same inverse power, this contribution of $j=0$ must vanish, so $\bar{F}_{j=0,\bar{w}}^u=0$. But then the second term in the brackets on the right-hand side of Eq. (B2) vanishes too, so

$$\bar{D}(f_{j=0}^u)=0. \tag{B3}$$

The same argument can now be applied to $j=1$ [because Eq. (B3) ensures that no contribution proportional to u^{-n-2} emerges from the term $j=0$], leading to $\bar{F}_{j=1,\bar{w}}^u=0$ and consequently $\bar{D}(f_{j=1}^u)=0$. Applying this argument term by term, we end up with the equation $\bar{D}(f_j^u)=0$ for all j, along with $\bar{F}_{j,\bar{w}}^u=0$. We express the general solution of this equation as

$$\bar{F}_j^u=a_j(\theta), \tag{B4}$$

where $a_j(\theta)$ is an arbitrary function of θ.

2. Case $w=v$

In this case the contribution of a particular j to Eq. (B1) is

$$\begin{aligned}\bar{D}(f_j^v)&=\bar{D}[R^{-iA-s}u^{-n-j}\bar{F}_j^v(\bar{w},\theta)]\\&=2\mu R^{-iA-s}\{(2iA+s)[\bar{w}\bar{F}_{j,\bar{w}}^v-(n+j)\bar{F}_j^v]u^{-n-j-1}\\&\quad+2\mu\bar{w}^2[-\bar{w}\bar{F}_{j,\bar{w}\bar{w}}^v+(n+j-1)\bar{F}_{j,\bar{w}}^v]u^{-n-j-2}\}.\end{aligned}$$

Again, for $j=0$ there is a contribution proportional to u^{-n-1}, which cannot be cancelled by any other j. Therefore, $\bar{F}_0^v$ must satisfy the differential equation $\bar{w}\bar{F}_{0,\bar{w}}^v=n\bar{F}_0^v$. Differentiating this equation with respect to $\bar{w}$, we find that the second term in the squared brackets vanishes too, so $\overline{D}(f_0^v)=0$. We can now repeat the argument for $j=1$, showing that $\overline{D}(f_1^v)=0$, then for $j=2$, etc. We find that for each j the demand that the contribution proportional to u^{-n-j-1} vanish implies

$$\overline{w}\overline{F}^v_{j,\overline{w}}=(n+j)\overline{F}^v_j, \tag{B5}$$

and when differentiating this equation with respect to $\overline{w}$, the contribution proportional to u^{-n-j-2} vanishes too, so $\overline{D}(f_j^v)=0$ for all j. The general solution of Eq. (B5) is

$$\overline{F}^v_j=\tilde{b}_j(\theta)\overline{w}^{n+j},$$

where $\tilde{b}_j(\theta)$ is an arbitrary function of θ.

APPENDIX C

We shall calculate here the coefficient α_v, describing the ratio between the leading-order tail amplitude at the CH and the initial tail amplitude at the EH for $s>0$. By virtue of Eqs. (109) and (93), we have

$$\alpha_v=\frac{b^{l_0}_{j=0}}{C_0}=\frac{b^{l_0}_{j=0}}{b^{+l_0}_{j=0}}. \tag{C1}$$

For $s>0$, Ψ is dominated at the two horizons by Ψ^v and Ψ^v_+. Similarly, the function $\psi^l_{k=0}$ is dominated at the two horizons by the second term on the right-hand side of Eq. (83) or (84):

$$\psi^l_{k=0}(r)\cong e^l_v R^{q_v} \quad (\mathrm{CH}),$$

$$\psi^l_{k=0}(r)\cong e^{+l}_v R_+^{q_v^+} \quad (\mathrm{EH}),$$

where $q_v=-iA-s$ and $q_v^+=iA-s$. It is convenient to rewrite these two asymptotic expressions as

$$\psi^l_{k=0}(r)\cong \tilde{e}^l_v\Delta^{-s}R^{-iA}=\tilde{e}^l_v\Delta^{-s}e^{im\Omega_- r^*} \quad (\mathrm{CH}) \tag{C2}$$

and

$$\psi^l_{k=0}(r)\cong \tilde{e}^{+l}_v\Delta^{-s}R_+^{iA}=\tilde{e}^{+l}_v\Delta^{-s}e^{im\Omega_+ r^*} \quad (\mathrm{EH}) \tag{C3}$$

[cf. Eqs. (34),(68)], with new coefficients $\tilde{e}^l_v$ and $\tilde{e}^{+l}_v$. Substitute now $v=t(1+r^*/t)$ into the asymptotic expressions (86) and (88) for Ψ^v and Ψ^v_+, collect terms of the same power of $1/t$, and compare the coefficient of t^{-n_0} to Eqs. (C2),(C3). One finds

$$b^l_{j=0}=\tilde{e}^l_v, \quad b^{+l}_{j=0}=\tilde{e}^{+l}_v. \tag{C4}$$

Therefore, in order to determine α_v we need the relation between $\tilde{e}^l_v$ and $\tilde{e}^{+l}_v$ (particularly for $l=l_0$). These two parameters can be obtained from the global $k=0$ solution (95). Comparing Eqs. (C2) and (97), we find, at the CH,[5]

$$\tilde{e}^l_v=B^l\lim_{r\to r_-}\left[R\,\frac{(1-x)}{1+x}\right]^{iA}=B^l\lim_{r\to r_-}[2R/(1+x)]^{iA}$$

[we have used here $F(a,b;c;0)\equiv 1$]. Similarly, from Eq. (C3) we find, at the EH,

$$\tilde{e}^{+l}_v=B^l\lim_{r\to r_+}[2R_+/(1-x)]^{-iA}$$
$$\times F(-s-l,-s+l+1;-s+1-2iA;1).$$

Equations (C1) and (C4) now yield

$$\alpha_v=\frac{\tilde{e}^{l_0}_v}{\tilde{e}^{+l_0}_v}=\Lambda^{iA}\Gamma, \tag{C5}$$

where

$$\Lambda\equiv\lim_{r\to r_-}[2R/(1+x)]\lim_{r\to r_+}[2R_+/(1-x)]$$

and

$$\Gamma\equiv 1/F(-s-l_0,-s+l_0+1;-s+1-2iA;1)$$
$$=\frac{\Gamma(-l_0-2iA)\Gamma(l_0+1-2iA)}{\Gamma(s-2iA)\Gamma(1-s-2iA)}.$$

Since A and Λ are real, Λ^{iA} is a pure phase factor. As it turns out, this factor depends on the choice of integration constant in the definition of $r^*(r)$, i.e., $dr^*=[(r^2+a^2)/\Delta]dr$. It is therefore not so clear whether this phase factor has much physical meaning. (One can relate this phase ambiguity to the fact that $\Omega_+\neq\Omega_-$.) With the convenient choice

$$r^*(r)=r+\frac{r_+^2+a^2}{r_+-r_-}\log|(r-r_+)/M|-\frac{r_-^2+a^2}{r_+-r_-}$$
$$\times\log|(r-r_-)/M|,$$

a direct calculation of $R\equiv e^{-2\kappa_- r^*}$ and $R_+(r)\equiv e^{2\kappa_+ r^*}$ yields

$$2R/(1+x)\cong e^{-\delta/2M}(\delta/M)^{-\delta/r_-} \quad (\mathrm{CH})$$

and

$$2R_+/(1-x)\cong e^{\delta/2M}(\delta/M)^{\delta/r_+} \quad (\mathrm{EH}),$$

so one obtains

$$\Lambda=(\delta/M)^{-\delta^2/a^2}. \tag{C6}$$

[5]This calculation holds for nonvanishing m. For $m=0$, Eq. (97) is ill defined and must be replaced by Eq. (100). One can show that in this case, too, $|\alpha_v|=1$. However, we do not need to calculate α_v for $m=0$, because the coefficient $C_0\alpha_v$ in Eq. (111) anyway vanishes in this case (as $C_0=0$).

We are primarily interested here in the absolute value of α_v. It is easy to verify that for any integers $l_0 \geqslant s > 0$ and real $A \neq 0$, $|\Gamma| = 1$. It therefore follows from Eq. (C5) that

$$|\alpha_v| = 1. \tag{C7}$$

APPENDIX D

In this appendix we analyze the term $k=1$ in the late-time expansion, for $s<0$, in order to find out whether the term $j=1$ in Eq. (85) vanishes or not.

We first summarize the results for the term $k=0$ (which serves as a source term for $k=1$). The function $\psi^l_{k=0}$ satisfies the homogeneous equation $D^l(\psi^l_0)=0$, whose general solution is given in Eq. (95). Equations (90), (91), and (105) imply

$$e_u^{+l} = A^l = 0 \tag{D1}$$

and, therefore,

$$\psi^l_{k=0} = B^l F^l_v(r) \quad (s<0). \tag{D2}$$

When we originally derived the relation $e_v^{+l} \propto B^l$ in Eq. (105), we restricted it to $s \geqslant 0$, because residual $O(\Delta)$ terms involved in the transition from $\hat{\psi}^u_+$ (the dominant component of $\hat{\psi}_+$ for $s<0$) to ψ^u_+ might overshadow the subdominant component ψ^v_+. However, Eq. (91) implies that the component $\hat{\psi}^u_+$ entirely vanishes for $s \leqslant 0$. Therefore, no such residual $O(\Delta)$ terms are present, and we can safely match $\hat{\psi}^v_+$ to ψ^v_+ (at the leading order in Δ). In particular, the matching of the term $k=0$ [i.e., matching $B^l F^l_v(r)$ in Eq. (95) to $e_v^{+l} R_+^{q_v^+}$ in Eq. (84)] implies $e_v^{+l} \propto B^l$ for $s<0$ too. Equations (90) and (93) now yield $B^l \propto C_0 \delta_{l,l_0}$, which we reexpress as

$$B^l = B^0 \delta_{l,l_0}, \tag{D3}$$

where B^0 is a nonvanishing parameter proportional to the initial perturbation at the EH:

$$B^0 \propto C_0. \tag{D4}$$

The functions $\psi^l_{k>0}$ satisfy the inhomogeneous equation

$$D^l(\psi^l_k) = S^l_k.$$

The general solution of this equation can be expressed as

$$\psi^l_k = A^l_k(r) F^l_u(r) + B^l_k(r) F^l_v(r), \tag{D5}$$

with

$$A^l_k(r) = \int^r \Delta(r')^{-2} S^l_k(r') F^l_v(r') W(r')^{-1} dr' \tag{D6}$$

and

$$B^l_k(r) = -\int^r \Delta(r')^{-2} S^l_k(r') F^l_u(r') W(r')^{-1} dr', \tag{D7}$$

where $\Delta(r') \equiv (r'-r_+)(r'-r_-)$ and W is the Wronskian of the two homogeneous solutions F^l_u, F^l_v. [The factor Δ^{-2} emerges from the factor Δ^2 which multiples ∂_{rr} in the differential operator D^l—cf. Eq. (80).]

We need to evaluate the asymptotic behavior of $\psi^l_{k=1}$ near the two horizons at order Δ^0. Let us first show that the term $B^l_k(r) F^l_v(r)$ in Eq. (D5) does not contribute at this order. From Eqs. (D2) and (81),(82) we have

$$S^l_{k=1} \propto F^l_v.$$

By virtue of Eqs. (99), (102), and (104), the integrand in Eq. (D7) behaves at both horizons like Δ^{-1}, so the integral diverges logarithmically. However, since F^l_v vanishes like Δ^{-s}, the term $B^l_{k=1}(r) F^l_v(r)$ will vanish at both horizons, and the entire contribution at order Δ^0 will come from the first term in Eq. (D5):

$$\psi^l_{k=1} \cong A^l_{k=1}(r_-) F^l_u(r) \quad (\mathrm{CH}). \tag{D8}$$

Similarly, at the EH the contribution at order Δ^0 is

$$\psi^l_{k=1} \cong A^l_{k=1}(r_+) F^l_u(r) \quad (\mathrm{EH}).$$

[As we shall show below, $A^l_{k=1}(r_+)$ vanishes, which actually means that the $O(\Delta^{|s|})$ contribution from $\hat{\psi}^{+v}_{k=1}$ dominates $\psi^l_{k=1}$ at the EH. However, the contribution (D8) at the CH does not vanish. We need to show the vanishing of $A^l_{k=1}(r_+)$ in order to obtain the integration constant in Eq. (D6).]

In the next step we relate the term $A^l_{k=1}(r) F^l_u(r)$ to the coefficients $a^{+l}_{j=1}$ and $a^l_{j=1}$ in the asymptotic expressions (85),(87). Equations (D1) and (106) imply $e^l_u = e^{+l}_u = 0$ for all l, and hence $e_u = e^+_u = 0$. At the CH, $e_u = 0$ yields $p^u_{k=1,i=1} = 0$ [as implied by property (3) in Sec. V], so $p^u_{k=1} = p^u_{k=1,i=0}$. Substituting this in Eq. (36), decomposing into spherical harmonics, and recalling that, for $s<0$, $\hat{\psi}^{v1}_k$ does not contribute at order Δ^0 near the CH, we obtain

$$\psi^l_{k=1} \cong \hat{\psi}^{ul}_{k=1} = p^{ul}_{10} R^{iA} \quad (\mathrm{CH}), \tag{D9}$$

where $p^{ul}_{10} \equiv p^{ul}_{k=1,i=0} = \mathrm{const}$. Matching this to the asymptotic expression (D8) [recalling the asymptotic form of $F^l_u(r)$ at the CH, Eq. (104)] yields

$$p^{ul}_{10} \propto A^l_{k=1}(r_-) \tag{D10}$$

(the proportionality factor in the last expression is a trivial nonvanishing constant). We now substitute $R^{iA} = e^{-im\Omega_- r^*}$ [cf. Eq. (34)] into Eq. (D9) and $u = -t(1 - r^*/t)$ and $a^l_{j=0} = 0$ into Eq. (85). Matching the order t^{-n_0-1} of the latter to Eq. (D9) yields $a^l_{j=1} = (-1)^{n_0+1} p^{ul}_{10}$. Combining this with Eq. (D10), we obtain the required relation at the CH:

$$a^{l}_{j=1} \propto A^{l}_{k=1}(r_-). \tag{D11}$$

Repeating the same matching procedure at the EH, one obtains the analogous relation

$$a^{+l}_{j=l} \propto A^{l}_{k=1}(r_+). \tag{D12}$$

(Recall that in the last two equations the proportionality factors are nonvanishing constants.)

The integration constant in Eq. (D6) is now determined from Eq. (D12). This equation, combined with the regularity condition (91), implies

$$0 = a^{+l}_{j=1} = A^{l}_{k=1}(r_+).$$

Therefore, in Eq. (D6) the lower limit of integration must be r_+. Defining $E^l \equiv A^l_{k=1}(r_-)$, we obtain

$$E^l = \int_{r_+}^{r_-} \Delta^{-2} S^l_{k=1} F^l_v W^{-1}\, dr, \tag{D13}$$

and Eq. (D11) reads

$$a^l_{j=1} \propto E^l. \tag{D14}$$

We turn now to evaluate the integral in Eq. (D13). In view of Eq. (81), we denote the contributions of $\overline{S^l_{k=1}}$ and $\hat{S}^l_{k=1}$ to E^l by $\overline{E^l}$ and $\hat{E}^l$, respectively:

$$\overline{E^l} = N^1_1 \int_{r_+}^{r_-} \Delta^{-2} \overline{T_1} \psi^l_{k=0} F^l_v W^{-1}\, dr$$

$$= \delta_{l,l_0} B^0 N^1_1 \int_{r_+}^{r_-} \Delta^{-2} \overline{T_1} (F^{l_0}_v)^2 W^{-1}\, dr \tag{D15}$$

and

$$\hat{E}^l = 2isaN^1_1 \sum_{l'} \lambda_{ll'} \int_{r_+}^{r_-} \Delta^{-1} \psi^{l'}_{k=0} F^l_v W^{-1}\, dr$$

$$= 2isaN^1_1 B^0 \lambda_{ll_0} \int_{r_+}^{r_-} \Delta^{-1} F^l_v F^{l_0}_v W^{-1}\, dr, \tag{D16}$$

where we have used $\psi^l_{k=0} = B^0 \delta_{l,l_0} F^l_v$, which follows from Eqs. (D2),(D3). (Clearly, $\tilde{S}^l_k$ and any term in S^l_k proportional to $\psi^{l'}_{k-2}$ do not contribute to E^l.) Note that the only possible nonvanishing $\hat{E}^l$ is for $l = l_0$ and $l = l_0 + 1$ (because λ_{ll_0} vanishes for $l > l_0 + 1$, and a mode $l < l_0$ does not exist).

So far we have considered all values of m, and we had $l_0 = \max(|m|,|s|)$ and $n_0 = 2l_0 + 3$. For a given s, the minimal value of l_0 (and n_0) is achieved for the m values $-|s| \leq m \leq |s|$, which yield

$$l_0 = |s|, \quad n_0 = 2|s| + 3.$$

We shall now restrict our attention to these values of m, as they dominate the overall late-time behavior in a generic situation.

We first calculate $\hat{E}^l$ for $l = l_0 + 1 = 1 - s$. In this case the last integrand in Eq. (D16) becomes

$$\left(\frac{1-x}{1+x}\right)^{2iA} [(1-s)x + 2iA] \Delta^{-s} \times \text{const.}$$

Its integral is elementary,

$$\left(\frac{1-x}{1+x}\right)^{2iA} \Delta^{1-s} \times \text{const.}$$

This function vanishes at both horizons, so $\hat{E}^{l_0+1} = 0$. Equation (D16) now becomes

$$\hat{E}^l = 2isaN^1_1 \delta_{l,l_0} B^0 \lambda_0 \int_{r_+}^{r_-} \Delta^{-1} (F^{l_0}_v)^2 W^{-1}\, dr, \tag{D17}$$

where $\lambda_0 \equiv \lambda_{l_0 l_0}$. Combining Eqs. (D15) and (D17), we obtain

$$E^l = \overline{E^l} + \hat{E}^l = \delta_{l,l_0} B^0 N^1_1 \int_{r_+}^{r_-} \Delta^{-2} \hat{T} (F^{l_0}_v)^2 W^{-1}\, dr,$$

where $\hat{T} \equiv \overline{T_1} + 2isa\lambda_0 \Delta$. We rewrite this expression as

$$E^l = \delta_{l,l_0} B^0 J \times \text{const}, \tag{D18}$$

where

$$J \equiv \int_{r_+}^{r_-} I(r)\, dr,$$

$$I \equiv \hat{T} \Delta^{s-1} (F^{l_0}_v)^2 = \hat{T} \Delta^{-s-1} \left(\frac{1-x}{1+x}\right)^{2iA},$$

and ‘‘const’’ denotes a trivial nonvanishing constant. [We have used here Eq. (99); recall that for the case $l_0 = -s$ considered here, the hypergeometric function in $F^{l_0}_v$ is 1.]

The calculation of the integral J is complicated. Our main goal, however, is to find out whether this integral vanishes or not. We shall use arguments of continuity (and analyticity) to show that J is generically nonzero. Let us denote the values of I and J for $a = 0$ (with fixed M) by I_0 and J_0, respectively. For $a = 0$,

$$\hat{T} = 2s(r\Delta - Mr^2) = 2sr^2(r - 3M),$$

so

$$I_0 = 2sr^2(r - 3M)[r(r - 2M)]^{-s-1}.$$

The integration of I_0 is elementary, and one finds that, for both $s = -1$ and $s = -2$, J_0 is nonzero [it is $-8M^4$ for $s = -1$ and $(32/3)M^6$ for $s = -2$]. Now, since I is a continuous function of a in the integration interval $r_- < r < r_+$, and since the integral J is absolutely convergent, J must be a continuous function of a. Therefore, there exists a range of values of a near $a = 0$ throughout which $J \neq 0$. Furthermore, since I is analytic in a in the interval $r_-<r<r_+$, J is also an analytic function of a. [For the present discussion it is useful to reexpress J as

$$J=-\frac{\delta}{2}\int_{-1}^{1}I(x)dx,$$

where $I(x)\equiv I(r(x),x)$ and $r(x)$ is the linear function obtained by converting Eq. (98). By this we eliminate the dependence of the integration limits on a. Again, this integral is absolutely convergent, and the integrand is analytic in a (for $|a|<M$) in the interval $-1<x<1$.] Since the analytic function $J(a)$ is nonvanishing in $a=0$ and its neighborhood, it must be nonvanishing everywhere in $|a|<M$, except perhaps at isolated values of a.

From straightforward dimensional considerations it is obvious that (for a given m and s) whether J vanishes or not can only depend on the dimensionless entity a/M. We conclude that J is generically nonvanishing, except perhaps for isolated values of a/M. (Furthermore, numerical calculations of the integral J for the modes of greatest physical interest, i.e., $l_0=-s=1,2$ and $s\leqslant m\leqslant -s$, and for various values of a/M, suggest that J never vanishes for these modes.)

Recalling now Eqs. (D14) and (D18), we conclude that $a^l_{j=1}$ vanishes for $l\neq l_0$, but is generically nonvanishing for $l=l_0$. We express this result as

$$a^l_{j=1}=C_0\alpha_u\delta_{l,l_0}, \tag{D19}$$

where the constant α_u is generically nonvanishing. [The coefficient C_0, denoting the initial amplitude of the mode $l=l_0$ at the EH, emerges from the parameter B^0 in Eq. (D18) via Eq. (D4).]

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